Continuous Quantitative Reasoning

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Abstract

This article addresses four primary questions: “What is continuous reasoning?" “Why is continuous reasoning important?" “How does someone reason both continuously and quantitatively?" and “How can we aid students in building continuous quantitative reasoning?" Experimental results (Bassock & Olseth, 1995; Castillo-Garsow, 2010) are used to highlight key distinctions in determining the meaning and role of continuous reasoning. These results are then placed in the context of a theoretical framework (P. W. Thompson, 2011, 2008a, 1990) that develops the meaning of continuous quantitative reasoning. The article closes by speculating on how these meanings of continuous reasoning and continuous quantitative reasoning might inform the future development of mathematics instruction.

Continuous Quantitative Reasoning

Covariation — the imagining of two quantities changing together — has been rising in prominence in mathematics education literature as a way of thinking about functions . (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994, 1995; Moore, 2010; Moore & Bowling, 2008; Rizzuti, 1991; Strom, 2008; Saldanha & Thompson, 1998; P. W. Thompson, 2008a, 2011). Covariational reasoning has been linked to student' understanding of geometric growth (Confrey & Smith, 1994, 1995; Strom, 2008), function (Carlson et al., 2002), trigonometry (Moore, 2010), calculus (P. W. Thompson, 1994b), and differential equations (Castillo-Garsow, 2010; Rasmussen, 2001).

However, the mechanics of covariation — imagining quantities and imagining change — are not well studied. The authors studying covariation have different images of what is involved in constructing an understanding of a continuous function via covariational reasoning. Rizzuti and Saldanha and Thompson describe initially continuous images of covariation. Confrey and Smith and Strom describe an image of covariation that begins discretely and becomes dense. Carlson et al. describe an image of covariation that begins with a simple awareness of change, later becomes discrete, and only after that becomes continuous. These different theoretical frameworks leave open questions of how students themselves come to understand ideas of continuity and continuous function.

This paper discusses two experimental results that are relevant to the question of how students build continuous functions. The first experiment, conducted by Bassock and Olseth (1995), was a statistical study in the transfer of continuous and discrete methods to different problem situations (primarily linear physics problems and arithmetic economics problems). The second experiment, conducted by Castillo-Garsow (2010) was a teaching experiment with two Algebra II students. These students constructed different solutions to a differential equation based on their different (discrete or continuous) understandings of change.

In reasoning covariationally about a continuous function, students must reason about quantity, about continuity, and about change. To begin the process of investigating how this might be done, I use the experimental results of Bassock and Olseth (1995) and Castillo-Garsow (2010) to address four primary questions: “What is continuous reasoning?" “Why is continuous reasoning important?" “How does someone reason both continuously and quantitatively?" and “How can we aid students in building continuous quantitative reasoning?"

What is Continuous Reasoning?

Before we can begin to discuss continuous quantitative reasoning, we must first identify what it means to reason continuously — or at least what I mean by continuous reasoning. Both “continuous" and “reasoning" are words that carry a great deal of nuance. A necessary distinction to make in this discussion is between a problem situation, the method used to solve it, and the reasoning that derives or selects that method.

Problem, Method, Reasoning

Bassock and Olseth (1995) studied cases of transfer of continuous and discrete models. Bassock and Olseth presented problems to students that described continuous changes in speed, or discrete changes in monetary investment. One group of participants learned to solve the physics problems with linear functions, and another group of participants learned to solve the economics problems with arithmetic sequences. Bassock and Olseth then studied the frequency of transfer of methods of one type (linear function, arithmetic sequence) to problems of the other type (discrete changing situation, continuous changing situation).

Bassock and Olseth classified problem situations into discrete or continuous (based primarily on the perception of the researchers), and classified methods into discrete or continuous. The continuous reasoning that I wish to talk about, however, is one level lower: the level of students’ reasoning that informs their choice of method. I will illustrate this distinction with an example from my own work. Consider the following problem, which was given to Algebra I students at a public high school:

> I'm trying to save up for a big screen TV. I make the decision to have \$55 of my monthly paycheck go towards the TV savings fund (previously my rainy day fund). After 4 paychecks, I have a total of \$540.
>
> Sketch a graph that shows how much money I have saved at each moment in time during the first 8 months after I make the change. Be sure to think about how much money I have saved *between* paychecks.

The student responses could be classified into three types, and I've included a "good" example of each type in the three figures below. In the tradition of cryptography, I've nicknamed the students "Alice," "Bob," and 'Carol."

In Figure 1, Alice began with \$55 at one month (or possibly \$0 at 0 months), and continued by plotting a new point each month, adding \$55 each time. There are some errors in this solution. Specifically, Alice did not account for the money (\$320) that was in the account when the big screen TV fund began, Furthermore, Alice did not follow the instructions to think about how much money was saved between paychecks. The graph has no value for non-integer numbers of months.

In Figure 2, Bob began with the correct \$320 at 0 months, and also plotted points. Bob's sketched triangles shows that he plotted these points by repeatedly incrementing the month by 1 and the account value by \$55. Once these values were plotted, Bob connected the dots to form a line. There is also an anomalous marked point, at (1/2, 374.5), corresponding to the amount of money the account would have at half a month if the account were earning money continuously rather than discretely. Bob's solution shows that he also did not think about the value of the account between paychecks, except perhaps as an afterthought. Bob's solution shows an account that is continuously earning money, so that the value of the account at 1.1 months is different than the value of the account at 1.2

months, which is not in keeping with an image of making a $55 deposit out of each (discrete) monthly paycheck. Bob's choice of a line shows that his solution was based on a purely discrete image of the situation, where he only attended to the mathematics of the situation once a month, and did not think about the situation at other times in-between months. The anomalous point at half a month is a way of complying with the instruction to think about the value of the account between month; however, since the student's work (partially cropped out in Figure 2) shows calculations for every point on the line except the half-month point, I suspect that that point was placed after the fact, based on the line, rather than before the fact, based on the situation. Looking at the work to the left, there are calculations for adding in increments of 55, but there are no calculations for finding or adding half of 55, which would have been necessary if the half-month point was model based.

Carol's solution (Figure 3) shows a similar approach to Bob's (Figure 2) in that she began at $320 and added $55 each month. However, Carol's solution differs from Bob's in that Carol's graph shows that the value of the account is not changing in-between monthly deposits[1]. Carol's solution showed that although the deposit events were discrete, the time that they were occurring in was continuous[2].

Now let us classify case according to Bassock and Olseth's model of discrete and continuous problems and methods. First in the classification of Bassock and Olseth, the problem is a discrete problem. The clock is driven by discrete events, and the problem is similar to the economics problems that Bassock and Olseth used. Alice's solution clearly

[1] Bob's graph does not show a constant account value in-between deposits, but it arguably also does not show a changing account value in-between deposits. Rather, it shows a lack of attention to the account value in-between deposits. Bob's line was drawn because of Bob's understanding that when there are dots on a graph, they should be connected with a line. The line is not an attempt at communicating something about account value.

[2] It is not clear from this solution if Carol drew the points first and then the steps, or if Carol drew the graph one step at a time. If available, that information would have shown which clock (discrete or continuous) Carol thought of as dominant.

shows a discrete method of repeatedly adding \$55. Bob's solution presents more of a problem. The solution itself is a continuous line, which would mean that Bob's solution is a continuous solution. However, the solution process itself was achieved by repeatedly adding \$55, which would be classified as a discrete method. Similar to Alice's solution, Carol also used a discrete method of repeatedly adding \$55.

However this classification of problem and method fails to capture the nuances of the three student's solutions, and it is here that we can make a third distinction between discrete and continuous *reasoning*. Both Alice (Figure 1) and Bob (Figure 2) show clear signs of discrete reasoning. They attended only to the discrete monthly events, and not to the continuous time that these events were occurring in. For this reason, Alice's solution makes no predictions for the value of the account between months, and Bob's solution makes incorrect predictions.

Although Carol's solution (Figure 3) is *not* continuous, the reasoning behind that solution *is* continuous. Carol's solution shows that she imagined the value of the account not just at deposit time, but also at every point in time during an 8-month period. This also shows a difference in perception of the problem situation. For Alice and Bob, the problem was a discrete problem, with an event driven clock. For Carol, the problem was a continuous problem, because time was passing continuously, and the events occurring in that continuous time just happened to result in a discontinuous function.

To summarize, Alice reasoned about the problem discretely and generated a discrete solution by a discrete method. Bob reasoned about the problem discretely and generated a continuous solution by a discrete method, Carol reasoned about the problem continuously and generated a discontinuous solution by a discrete method. These examples show that continuous or discrete reasoning is distinct from solution shown and from the method used.

Chunky and Smooth

Now that we've identified continuous reasoning as distinct from both a continuous solution and a continuous method, we can investigate the nature of the mental actions involved in continuous reasoning. For this discussion, I will draw on results from a teaching experiment involving two high school Algebra II students ("Derek" and "Tiffany") from a public high school (Castillo-Garsow, 2010). In order to build an understanding of exponential growth, the students were given a number of financial modeling tasks. Following the methodology of Steffe and Thompson (2000), the full teaching experiment itself covered 15 teaching episodes of 50 minutes each, with five participants (students Derek and Tiffany, teacher-researcher Carlos, observer Pat, and a videographer) The full details of this teaching experiment are too lengthy to include here, but are available in my dissertation (Castillo-Garsow, 2010). For this example, I'll discuss their work with the first task, a simple interest bank policy.

> Jodan bank uses a simple interest policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of eight percent of the initial investment per year.

During the discussion of this policy, I asked the students to imagine that a person named Patricia had invested $500 into an EZ8 bank account. The students were asked a number of questions about the behavior of Patricia's account, and in the course of discussion they created this graph of Patricia's account value over time (Figure 4).

During a later episode of the teaching experiment, I revisited this graph, asking the students: "Explain to me what you're seeing. What does this graph show?" The students' responses are below.

> *Tiffany*: This graph shows that his... It starts at five hundred, because at zero years it starts at five hundred, and then it increases little by little every year... and not even every year but every like parts of it, just little by little. Like,

umm, even every day of the year. That's a that's a part of a year. So even every day of a year it's still growing tiny, tiny bit.

Tiffany's quote describes a change process that is occurring in completed "chunks." The change occurs year-by-year, or day-by-day. In each case, Tiffany is imagining that change occurs when a certain amount of time has passed, and a certain amount of change is associated with that time. Although the year-by-year changes are occurring within Tiffany's personal time as she experiencies it, Tiffany does not imagine change occurring within the year as time moves for her personally — instead she imagines time out of sequence with the events within a year not imagined until after she has imagined that the year is complete. Tiffany does not imagine that change is occurring within that time chunk unless she re-conceptualizes the change to a smaller chunk size, by imagining that the completed year has been chopped up into day sized chunks, each with its own associated completed change.

Another example of chunky thinking can be seen in the following two exchanges with the teaching experiment observer, Pat.

Pat: If I'm going sixty-five miles per hour what does that mean?

Tiffany:That in one hour you've gone, you should have gone sixty-five miles.

In this exchange Tiffany shows how she thinks of a rate of sixty-five miles per hour. She imagines a completed chunk of one hour and corresponding chunk of sixty-five miles. Or to put it differently, Tiffany imagines that Pat *has traveled* for one hour and for sixty-five miles[3]. This leads her into difficulty in the next exchange:

Pat: Can I travel for just one second at sixty-five miles per hour?

[3]This is similar if not identical to the concept "speed-length" in which a speed is imaginined as a solid length (like a ruler) repeatedly laid end-to-end, with the number of repetitions serving to keep track of time. (P. W. Thompson, 1994a; P. W. Thompson & Thompson, 1994; A. G. Thompson & Thompson, 1996)

> *Tiffany*: No. You have to do... You would have to do, um... Well, yeah, you could.

When Pat asks if he could travel for sixty-five miles per hour for just one second, Tiffany's first response is "no," because she has already imagined a completed chunk of one hour. She then re-conceptualizes that hour as being composed of smaller "one second" chunks and then changes her response. However just as initially there was nothing smaller than the one hour chunk, once she has re-conceptualized her chunk size as one-second, there is nothing smaller than a one second chunk unless she re-conceptualizes one second as being composed of smaller, (temporarily) atomic chunks. This process of chopping up a chunk to make smaller chunks requires noticeable time and conceptual effort on the part of Tiffany.

For a contrast, let's turn to Derek's description of the graph in Figure 4.

> Derek: It's growing constantly, but once it gets to one year, it's a total of eight percent higher. And then it grows by still eight percent higher than the five hundred, but just takes that value and its gets up to there each year. Carlos: OK, so what do you mean by it's growing constantly? Derek: It's always more money is being put in, because... and keep going.

Here, Derek is describing a very different process. Derek imagines that the account *is growing* in the present tense. That is Derek imagines that there is a mapping from his own current experiential time to a year in Patricia's world, and that as he is talking, time "gets to one year" and the account is growing (in experiential time) until it reaches 8% of $500. At the end of this process there is a completed chunk of one year and $40. This differs from Tiffany's chunky thinking in that there is a present-time experiential process going on *within* that one year *prior* to the one year chunk being completed. To contrast this with chunky thinking, I refer to this act of imagining a change in progress as "smooth thinking."

Chunky and Smooth in Continuous Reasoning

Here we have classified two different ways of thinking about change: thinking about change in completed chunks (chunky thinking), or thinking about change as a change in progress (smooth thinking). How do these two ways of thinking about change relate to continuous reasoning?

Chunky thinking is inherently discrete. It remains an open question whether or not continuous understanding can be built from pure chunky thinking. If continuous reasoning can be built from pure chunky thinking, however, it certainly has a high conceptual overhead. Chunky understanding prefers integers (number of chunks), and can (though cutting up) achieve change on the rational numbers, but change in the real numbers (required for continuity) is more difficult. I suspect that a chunky understanding of change in the real numbers cannot be achieved without at least some of the tools of modern analysis — specifically the formal definition of limit needed to construct the real numbers from the rational numbers.

Smooth thinking, in contrast, is inherently continuous. By imagining change in progress, change is subjected to that person's understanding of change in the physical universe. When Derek imagined change in progress from zero to one year, he could not imagine jumping directly to one year, because in order for time to get to one year, it has to pass through every moment of time before that year. Similarly, if one is imagining a car sixty-five miles in one hour as a change in progress, then car cannot reach sixty-give miles without passing through every distance in-between, and time cannot reach one hour without passing through every moment in-between.

The Different Roles of Chunky and Smooth

When working with linear functions, there is very little difference in the conclusions reached by students who begin with a completed chunk and divide it up and students who reach that chunk through a process of imagining smooth change in progress. When dealing with non-linear functions (such as the step function solution in Figure 3) smooth thinking becomes more important. When dealing with ideas of calculus, and specifically exponential growth, smooth thinking becomes critical. This is illustrated in the result from a later part of the same teaching experiment with Derek and Tiffany (Castillo-Garsow, 2010).

The Problem

During the teaching experiment, the students had been given a bank account policy which described the rate of change of the bank account as proportional to the (current) value of the account (Castillo-Garsow, 2010). After creating an equation $f(x) = 0.08x$ relating the rate of change of the account in dollars per year $f(x)$ to the amount of money in the account in dollars x, the students had been individually asked to create a graph of this policy from their equation, and both students created graphs like the one in Figure 5. Specifically, they were asked to create a graph of the rate of change of the account on the vertical axis and the value of the account on the horizontal axis. This style of parametric graphing — in which time is implicit — is called the "phase plane" in dynamical systems.

In Figure 5, Tiffany created a graph of the phase plane behavior of a bank account. specifically, an account where the rate of change of the account (measured in dollars per year) is a function (8%) of the value of the account (measured in dollars). This is illustrated by the marked points on the graph: \$500 corresponds to a rate of change of \$40/yr.

In separate interviews, the students were each asked to use the phase plane graph they had created to construct a graph of the behavior of the account over time. Specifically, I asked each student to graph the first two seconds of a back account under this policy, beginning with an investment of \$500. The student's approaches to this problem were very different.

Tiffany. Tiffany's approach to this problem was based on completed chunks. Tiffany began by imagining that the account began with \$500 at time 0, and that there was a rate associated with that \$500, based on the graph. Then after one second passed, that rate determined a corresponding amount of change and a new (larger) account value, and therefore a new (larger) rate. After the next second, the new (larger) rate would determine a larger amount of change, and a new account value. Tiffany was able to continue this pattern, imagining that each subsequent second would generate a larger change than the previous second (Figure 6). This approach, of fixing the rate over a small interval and then recalculating the rate from the new amount at the end of the interval bears a strong similarity to compound interest (Castillo-Garsow, 2010; P. W. Thompson, 2008a) as well as to Euler's method of solving a differential equation.

When I asked Tiffany how to fill in what goes on in-between points, Tiffany repeated the process on a smaller scale, filling in point by point, and describing a process of using each point to find the next point. As before, she plotted the points so that each change was larger than the change immediately before it (Figure 7). While she was plotting these points, she described a process of using the (hypothetical) value of each point to find the next point.

Tiffany had a sense that each point needed to be higher than the previous point, and each change needed to be higher than the previous change, but that only gave her information about the relationship between neighboring points, not the overall curve. This can be seen in Figure 7. Tiffany's attempt at filling in the values for one second took the form for two sequences of points, one sequence near zero seconds and one sequence near one second. Tiffany does not draw the remaining points needed to connect the two sequences, because she has only local information about neighboring points, and cannot visualize enough points at once to place them all in such a way that each change would be greater than the previous change while still connecting with the points at zero seconds and one second.

Note also that the two new sequences in Figure 7 were not drawn in such a way that they would connect with each other, even if Tiffany had continued them. The sequence at zero seconds curves up two quickly. If the first sequence pattern were to continue, the value predicted by the sequence at one second would be too high. Similarly, the sequence at one second is too shallow, if the second sequence pattern were continued backwards, the value predicted by the sequence at zero seconds would be too high.

Tiffany was unable to give a description of what the function would look like if enough points were filled in that the individual points were no longer visible, only saying that it would look like a solid line. When I asked her if she meant a straight line, she agreed.

Derek. Similar to Tiffany, Derek was asked seperately to construct an equation of the per-capita policy and graph that equation, generating a phase plane graph. Later, he was also asked to recreate the graph of the account over time from the phase plane graph. Derek's approach to the task of reconstrucing the time graph from the phase plane graph he had created was based on imagining changes in progress. Derek's approach was extremely quick and intuitive, and he jumped immediately to his conclusions while saying very little to describe his thought process. Here I've reconstructed his second of mental actions based on his conclusions and body motions.

> *Carlos*: So can you show me how the money in your ... in your account is growing, umm.
>
> *Derek*: On that axis?
>
> *Carlos*: By moving your finger along this axis, yeah.
>
> *Derek*: Like starts slow and then just keeps getting faster and faster. *Carlos*: OK umm and what about the rate of growth? 6 *Derek*: It would also start slow and keep getting faster and faster.

Derek imagined that as time was passing for him, time was also passing for the account (although much slower, in that he only imagined the first two seconds). Since the rate was always positive, Derek knew that as time was passing for the account, the value of the account was growing, Derek also knew (from the phase plane graph), that as the value of the account was growing, the rate of growth of the value of the account was also growing, so the account value was growing faster and faster. Lastly, Derek knew that since the account was growing faster and faster, the rate of growth of the account (as an increasing function of account value), would also have to be growing faster and faster. These conclusions led Derek to create the graph in Figure 8.

Discussion

Derek's method gave him the overall behavior of the curve, but it is also a purely qualitative method. Derek made no numerical predictions about the value of the curve at any time other than the initial value, and he would have been unable to. In imagining changes in progress, there is no value that ever stays constant long enough to be calculated. In order to calculate the value of a function $f(x)$ from an x, the mathematical operations of addition, multiplication, etc. involved in finding $f(x)$ from x take experiential time to carry out. If x is changing in that experiential time, then before the first mathematical operation has begun, the value of x has changed, and calculating values of $f(x)$ becomes a problem of hitting a moving target. To borrow from Breidenbach, Dubinsky, Hawks, and Nichols's terminology (1992), an x and $f(x)$ changing in experiential time requires a process view of function, however, a process view of function involves imagining that a calculation is possible, *not* carrying out the calculation. Carrying out the calculation to find $f(x)$ from x requires an action approach that is incompatible with experientially changing x.

In contrast, Tiffany's method is quite capable of giving numerical approximations of function, and Euler's method (to which Tiffany's method bears a strong resemblance), is capable of approximating the solution to systems of much more complex differential equations. However, Tiffany's approach had the disadvantage that it did not give the overall behavior of the function as easily as Derek's method. In most cases, mathematicians simply "connect the dots" because we know enough about how these classes of functions behave to know that connecting the dots is acceptable. Tiffany could not make that same leap, because she had seen enough badly behaved functions to be cautious, but did not have enough experience to know that this function was well behaved.

Although it is possible to calculate enough points to determine the behavior of the function using Tiffany's method alone, this is time consuming and difficult. The advantage of Derek's method is that it gave the overall behavior of the curve quickly and intuitively.

Continuous Quantitative Reasoning

Smooth reasoning is inherently continuous, but its volatile nature makes it extremely difficult to engage in numerical calculations during a change in progress. As a result, smooth reasoning is far better suited for qualitative, rather than numerical reasoning. So what might be the mental actions involved in continuous *quantitative* reasoning? Part of the difficulty in determining a meaning for continuous quantitative reasoning is that "quantitative" has two meanings, both of which are important: "Quantitative" in the sciences refers to the idea of measurement, while "quantitative" in mathematics refers to numerical results, but not necessarily the idea of measurement. Similarly, covariation literature includes examples of covariation both with and without measurement (Carlson et al., 2002; Confrey & Smith, 1994, 1995; P. W. Thompson, 1990; P. W. Thompson & Thompson, 1994; A. G. Thompson & Thompson, 1996). The work of Thompson (P. W. Thompson, 1990, 2008b, 2011) addresses both of these meanings and provides two possible approaches to continuous quantitative reasoning.

Measurement and Quantitative Reasoning

P. W. Thompson (1990, p. 5) defines a quantity as "a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit." In this case, the aspect of P. W. Thompson's definition I wish to focus on is the measurement process. Part of quantitative reasoning is imagining an act of measurement that results in a number of some unit. Imagining this act of measurement can be difficult if the result of the measurement is always changing.

Two possible solutions to imagining smooth reasoning as a measurement process are possible: One is to imagine a measurement process that is itself well suited to continuous variation, so rather than imagine measuring distance as a result of laying rulers end to end and counting them, one could imagine measuring distance by unrolling a tape-measure, so that as the distance changes, the tape measure continues to unroll, and always reads the correct number. Similarly, time can be imagined as a changing value on a running infinite-precision stopwatch. However, this approach has the disadvantage that it obscures the role of the unit in the measurement process. The measurement is simply a reported number, not the result of a multiplicative relationship between a unit and a quality (the distance is some many times as long as the length of a foot).

A second alternative is to conceive of the measurement action once, and then to imagine that that action could be carried out for each value of the currently changing quantity, without actually imagining the steps involved in taking that measurement. To borrow from APOS theory (Breidenbach et al., 1992) this would be the construction of an (instantaneous) process conception of measurement from an action conception of measurement, enabling the student to imagine the infinite number of measurements needed to follow a continuously changing quantity.

Mixing Chunky and Smooth Reasoning

Both the options above are quantitative in the sense that they involve measurement, but they are not quantitative in the more traditional mathematical sense. We still do not have a way to predict numerical values of a quantity from smooth reasoning. Any numerical values associated with smooth reasoning must come from an outside source, either a measurement instrument (the tape measure and the stopwatch), or something like a computer that outputs measurements quickly enough that they appear to be changing in real time.

A method that does not rely on external tools is to make use of chunky reasoning to supply the numerical values. One could imagine that a smooth change in progress is occurring over an interval of time, and then at the end of that interval, the change has been completed, and can now be measured, resulting in a numerical value. Then one imagined smooth change in progress occurring in the next interval, and at the end of that interval another completed chunk and another measurement, and so on.

P. W. Thompson (2008b, 2011) describes a more sophisticated version of this process, a recursive way of thinking which he calls “continuous variation." In continuous variation, every smooth change in progress is imagined to be composed of smaller chunks (giving numerical values), and every small chunk within the change in progress is thought of as being itself covered by a smooth change in progress. So that the student achieves infinite precision by alternating smooth and chunky thinking as he or she chops the interval of variation into finer and finer chunks.

We have now arrived at an informal meaning of continuous quantitative reasoning, as repeated process of imagining the smooth change in progress of a quantity over an interval, followed by an actual or imagined numerical measurement of the quantity at the end of each interval.

Building Continuous Quantitative Reasoning

How does one build an understanding of continuous quantitative reasoning? Doing so requires that the student develop an image of measurement, as well as strong smooth *and* chunky reasoning abilities.

However, it may be that the best way to develop both smooth and chunky reasoning is to focus on instruction in smooth reasoning. Tiffany, who used only chunky reasoning, had great difficulty in situations that called for continuous reasoning. With one exception, she never engaged in smooth reasoning in the nine teaching episodes that she participated in. Derek, in contrast, showed great facility with both chunky and smooth reasoning. He was able to follow Tiffany’s reasoning, and on occasions when they were separated, he duplicated Tiffany's chunky results in addition to the results he developed based on smooth reasoning.

It makes sense that it would be easier to go from smooth to chunky than from chunky to smooth. An epistemic student attempting to reason continuously from chunky thinking can mimic the appearance of smooth thinking by making the chunk size very small, but as Tiffany's 'two second' solution to the phase plane problem (Figure 6, Figure 7) shows, no matter how small the chunks get, conceptually, there are always gaps where nothing is imagined going on between the chunks. So that chunky thinking can mimic the appearance of smooth thinking, but never achieve smooth thinking.

In contrast, smooth thinking very nearly implies chunky thinking for free. Smooth thinking involves imagining a change in progress, however a student cannot imagine a change in progress forever. The student must anticipate that at some point, that imagining process must end, at which point the student no longer has a change in progress, but now has a completed change — a chunk. Even within a change in progress, chunks can be generated with very little conceptual overhead. By anticipating smooth reasoning over several chunks, the student may temporarily pause a change in progress to achieve a completed chunk, and then continue the change in progress once reasoning on the chunk is complete.

So if smooth thinking implies chunky thinking, then education targeting continuous quantitative reasoning should focus on developing smooth reasoning skills. This cannot happen merely at the level of presentation and lecture, but must be done carefully and with attention to every aspect of the process. For example, in a study of graders of college algebra courses, Ye (2011) obtained student graphs and grading data that suggest that the grading methods used to cope with large classes and high volume grading may reinforce discrete thinking.

The Role of Technology

One possible way to target the development of smooth reasoning is to take advantage of the fact that outside of math class, students make use of smooth reasoning every day. As students walk, drive, live, and move, they are always experiencing changes in progress, and these changes are changes in a continuous world. People learn in infancy that they cannot pass from one point to another without passing through every point on the way, and similarly that one cannot simply leap into the future without passing through every moment of time between now and then. Continuous change in progress is part of our every day lives, and it is something that, on the sensorimotor level, students understand very well. Otherwise they would attempt to walk through walls much more often.

In early math classes, however, discrete reasoning takes precedence. We teach children to count, to do calculations, to find numerical values and to plot points to find a graph, all of which require stopped change, completed change. This is due in part to the nature of the numerical fluency we wish to teach students in the early grades; however the prevalence of discrete thinking (especially in later grades) is also due to the limitations of the technology currently used. Diagrams on paper can show only static images, not images of continuous change. Calculations on paper are time consuming and require stopping an imagined change. In both cases, a student's natural sense of how things move and change in their physical experience is lost.

Modern technology may provide representations that are better suited to continuous quantitative reasoning. Computers are capable of displaying animations that occur within a student's experiential time, helping them to imagine continuous change of both time and other quantities that can be measured in the animation. Computers can also label these animations with numerical values. By performing extremely rapid calculations, computers can create the illusion of continuously changing numbers, helping students to build an instantaneous, process conception of measurement. These types of animations can serve as didactic objects (P. W. Thompson, 2002) that support rich mathematical conversations about the behavior of and the relationships between the changing quantities depicted by the animation. Although an animation by itself will not move the student, the combination of animation and carefully constructed conversation enhances student opportunities to reflect on what they view, enabling the instructor to develop student understanding towards a goal.

Lastly, with the introduction of the low-cost touch tablet, it may be possible to engage student's understanding of continuity at a more physical level. Students already have an understanding of continuity in their own actions, and it is possible that well designed interactive animations can tap into these physical motions to build strong mathematical understandings of continuity. Interactive animations in which students drag objects with their fingers replicate the behavior of objects in the real world. Just as a finger must pass through every point on its path, the object being dragged must also pass through every point on its path, creating an inherent physical restriction that enforces continuous reasoning. In the context of conversation that emphasizes quantity, measurement, and magnitude, these types of physical activities could then support reflecting abstraction that develops a more abstract understanding of the mathematical principles the animation was intended to illustrate.[4]

The role that technology can play in the development of continuous quantitative reasoning is still very much an open problem and one that I look forward to investigating in the future.

Conclusion

Continuous quantitative reasoning is the coordination of two types of reasoning: continuous reasoning, and quantitative reasoning. Although it may be possible to arrive at continuous reasoning from discrete reasoning, it seems that continuous reasoning has at its

[4] An early example of what such interactive animations might look like — as well as some of the conversations that might be had around them — is available in the appendix. Additional examples are available on my website `http://www.math.ksu.edu/~cwcg/demo`

core the imagining of a change in progress, of the physical experience of continuous change based on real world motion in space and/or time (smooth reasoning). This type of smooth reasoning exists in contrast to imagining change as occurring in completed chunks (chunky reasoning). In chunky reasoning change is first conceived of at the end of the chunk, and only later (or not at all) conceived of within the chunk, which disrupts the process of imagining change in real time.

The results of the teaching experiment (Castillo-Garsow, 2010) show that smooth reasoning intuitively introduces the idea of passing through every point in-between the beginning and ending of a change without the much more sophisticated limit reasoning required by a discrete only approach. This intuition enables students to solve problems that would otherwise require much more elaborate mathematical toolkits.

Bassock and Olseth (1995) found that students were far more likely to use a discrete method for a continuous problem than they were to use a continuous method for a discrete problem. They attributed this assymetry in transfer to the constraints of continuous structure: that applying a continuous method to a discrete problem would fill in gaps that do not necessarily exist. This explaination attributes continuous reasoning to students — it implies that students are reluctant to fill in the gaps because they notice that there are gaps. The example of Alice, Bob and Carol suggests a second possibility: that students are more likely to use a discrete method because they are reasoning discretely, and a discrete method is what is generated by that discrete reasoning. The example of Bob (Figure 2) shows how discrete reasoning can result in a continuous solution, while the example of Carol (Figure 3) shows how continuous reasoning can result in a discontinuous solution.

The extent to which students engage in quantiative reasoning (as defined here) depends in part on the goals of the teacher and the types of problems the students are faced with. In scientific or modeling based instruction, quantitative reasoning may refer to the idea of measurement, while in abstract mathematics — where measurement is impossible — quantitative reasoning may refer to numerical or analytic solutions.

The juxtaposition of continuous and quantitative reasoning is difficult. Without infinite precision and instantaneous calculation, imagining a numerical result of a measurement or calculation requires that the process of change be paused, which ends smooth reasoning and begins chunky reasoning. Two possible solutions have been suggested: P. W. Thompson (2008b, 2011) suggests an image of “continuous variation" in which the student anticipates the possibility to alternate between smooth and chunky reasoning in smaller and smaller nested intervals in order to obtain the precision that the student believes is necessary for the problem — essentially introducing the idea of cover sets. Alternatively, technology, by providing the illusion of instantaneous calculation and the (not quite as convincing) illusion of infinite precision may aid students in building a process conception of numerical measurement and calculation compatible with smooth reasoning. These two solutions are not incompatible, and it may be that a combination of both is what best enables students to build continuous quantitative reasoning.

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Appendix: Sample Prototype of Interactive Animation

Many students initially learn slope, or "go over one and up the value of the slope *m*." In high school algebra classes, where students are only asked about the slope of linear functions, this understanding is sufficient to solve the problems at hand. In order to make sense of derivative as a limit, students need a concept image of slope over a continuous domain: a constant ratio between two changing quantities Δy and Δx. In order to address the distinction between these two ways of thinking, students need to work with non-linear or discontinuous functions. The "meaning of slope" prototype[5] is a series of three activities designed to address this distinction in an accessible way.

In the first activity, students are presented with a graph of a line with a slope of 2, and a right triangle with sides 1 and 2 that moves along the line. Students can drag the triangle and see that going "over 1 and up 2" will touch every point on the line. The

[5] Found at `http://www.math.ksu.edu/~cwcg/demo`

students then repeat this activity with two discontinuous functions (Figure 9), designed so that the triangle also touches every point on those functions, while retaining side lengths 1 and 2. The discussion around this activity covers the idea that “over 1 up m" is not enough to define slope, because they have seen examples of functions with zero and negative slope that still fit the “over 1 up 2" triangle.

In the second activity, students are again presented with a line, but this time the students have the option to resize the triangle. The students can adjust the horizontal Δx leg to any size, and the vertical Δy adjusts automatically to fit. The discussion for this activity centers around the idea that Δx can be thought of as a changing continuous quantity, and Δy as a function of Δx. When the equation $\Delta y = m\Delta x$ holds for every Δx, then they have a line.

The third activity returns to format of the original, with a triangle that can be dragged. In this version, however, the students have a field where they can change the size of the triangle’s Δx leg. Students are presented with a variety of functions that all fit “over 1 and up 2" and asked to find a Δx value that will “break" the fit by marking a point not on the original function. Students will be able to “break" every function except the line, establishing that a line is the only type of function that has the property $\Delta y = m\Delta x$ holds for every Δx. The discussion around this activity will help students see that this property is a better fit for students’ intuitive sense of slope.

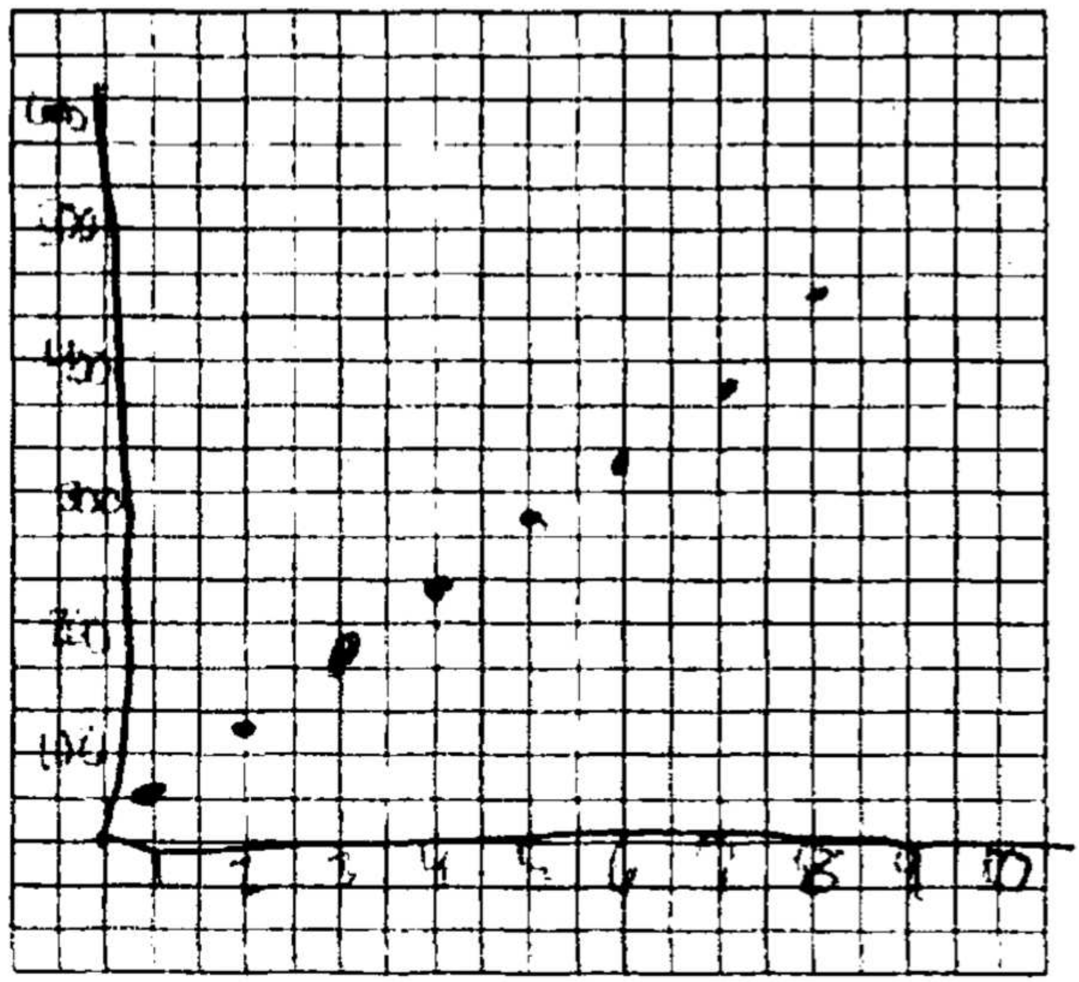

Figure 1. Alice solved the big screen TV problem by plotting discrete points.

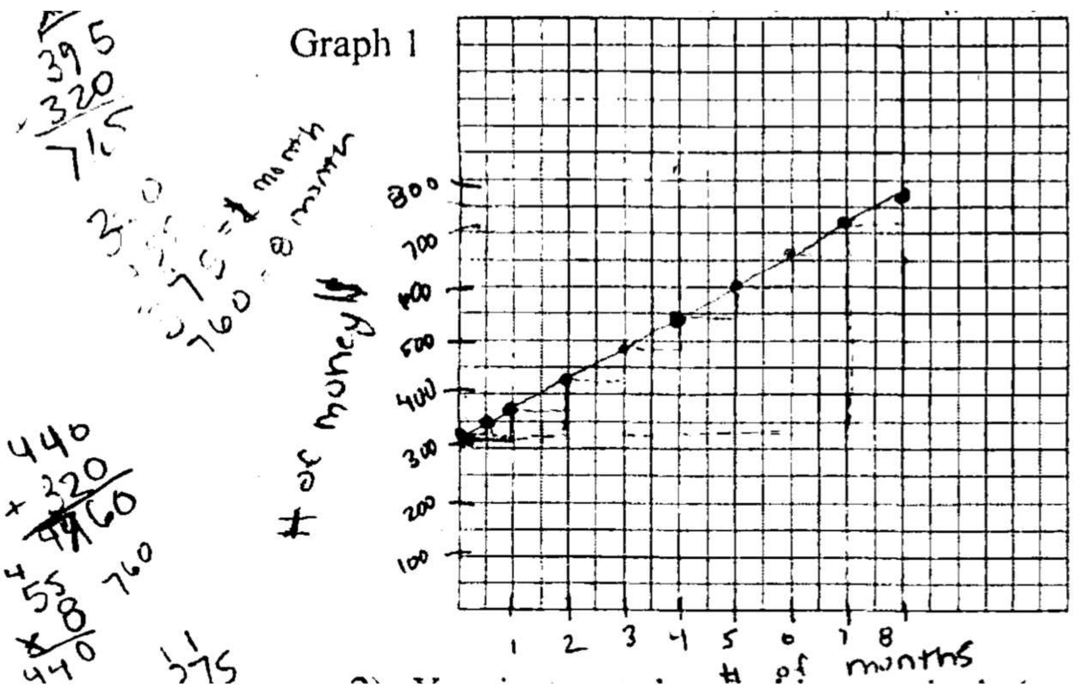

Figure 2. Bob solved the big screen TV problem by plotting points and drawing a line.

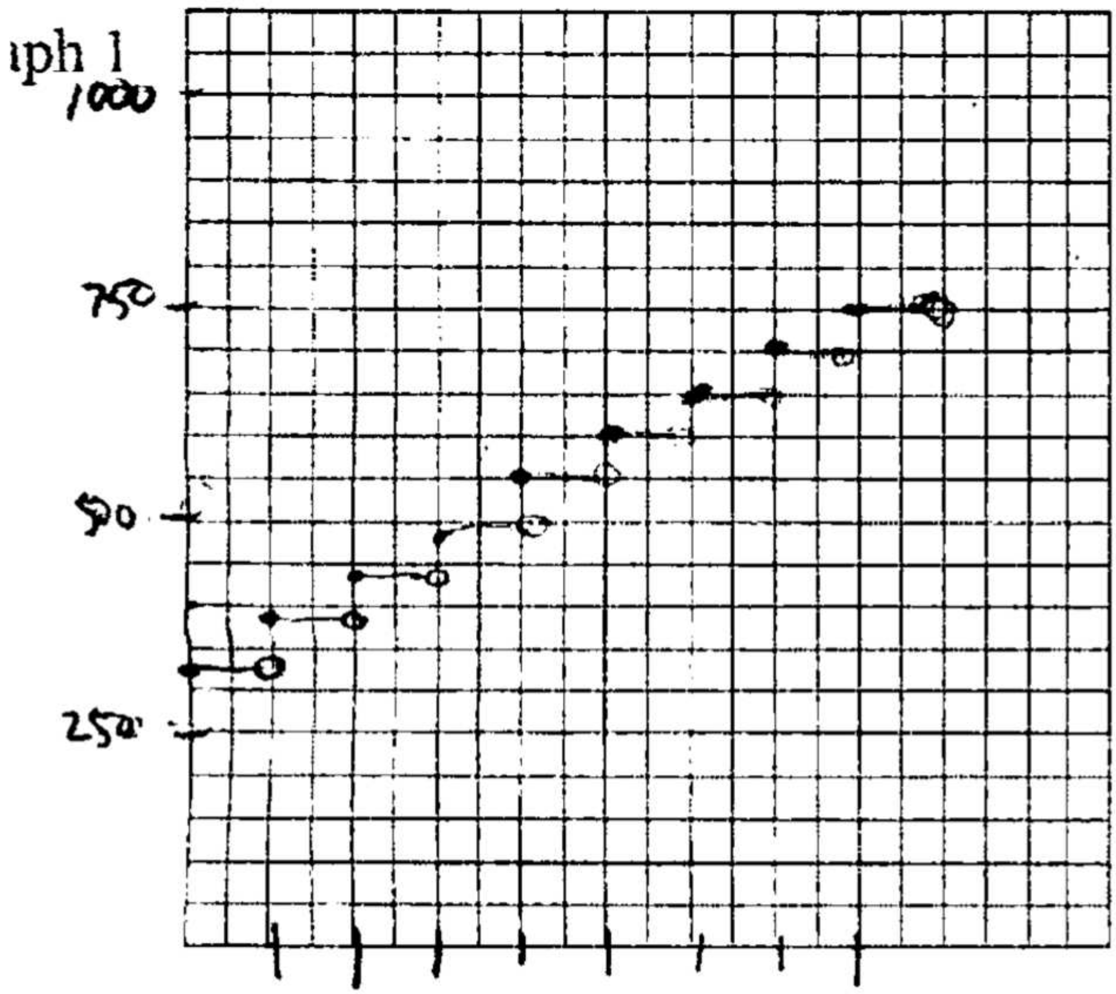

Figure 3. Carol solved the big screen TV problem by repeatedly adding $55 each month, and creates a step function to show that the value of the account does not change between months.

Figure 4. Derek and Tiffany's graph of Patricia's account value over time. I've added the unit labels on the axes after the fact for readability.

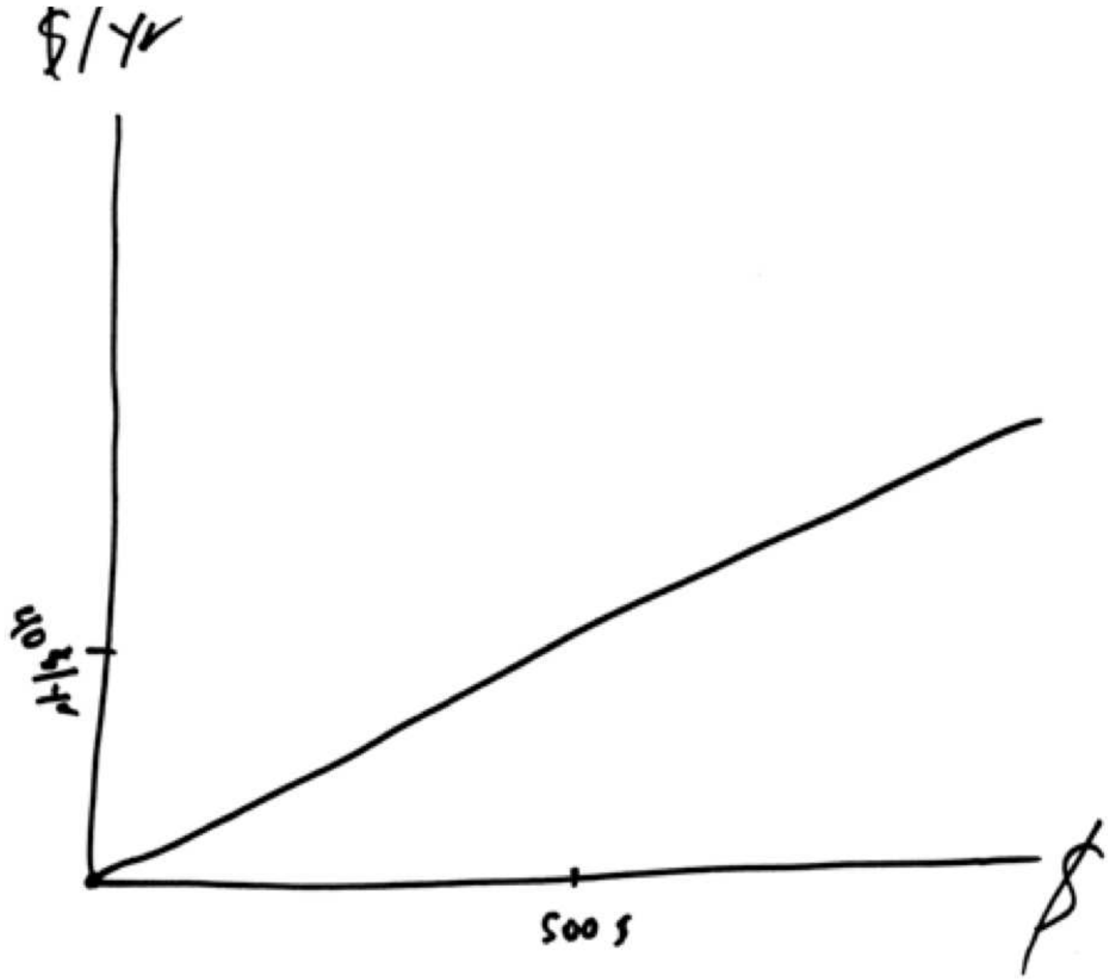

Figure 5. Tiffany's graph of a rate proportional to amount. In this case, the constant of proportionality was 0.08.

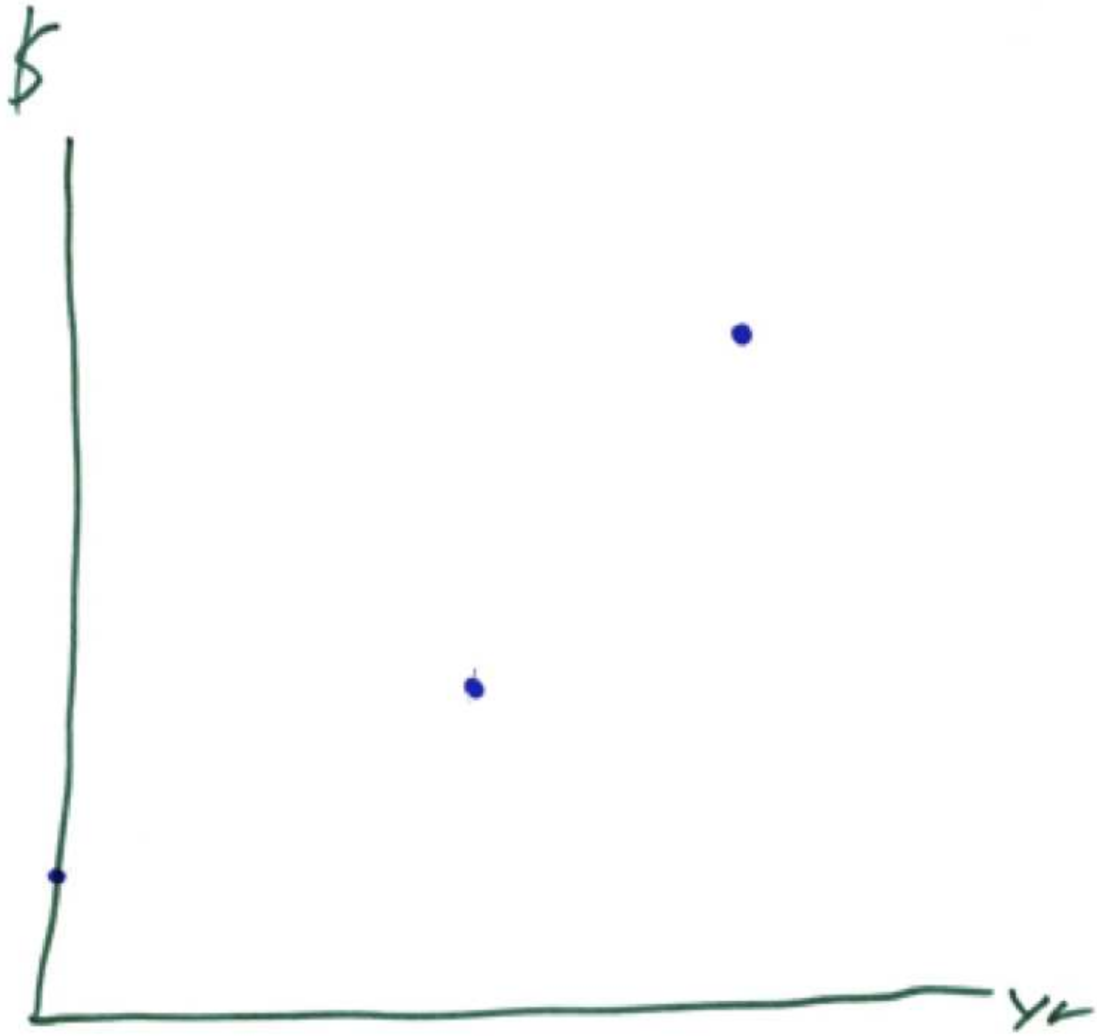

Figure 6. Tiffany's graph of the first two seconds of the account. Although the horizontal axis reads years, each point occurs one second (a tiny fraction of a year) apart.

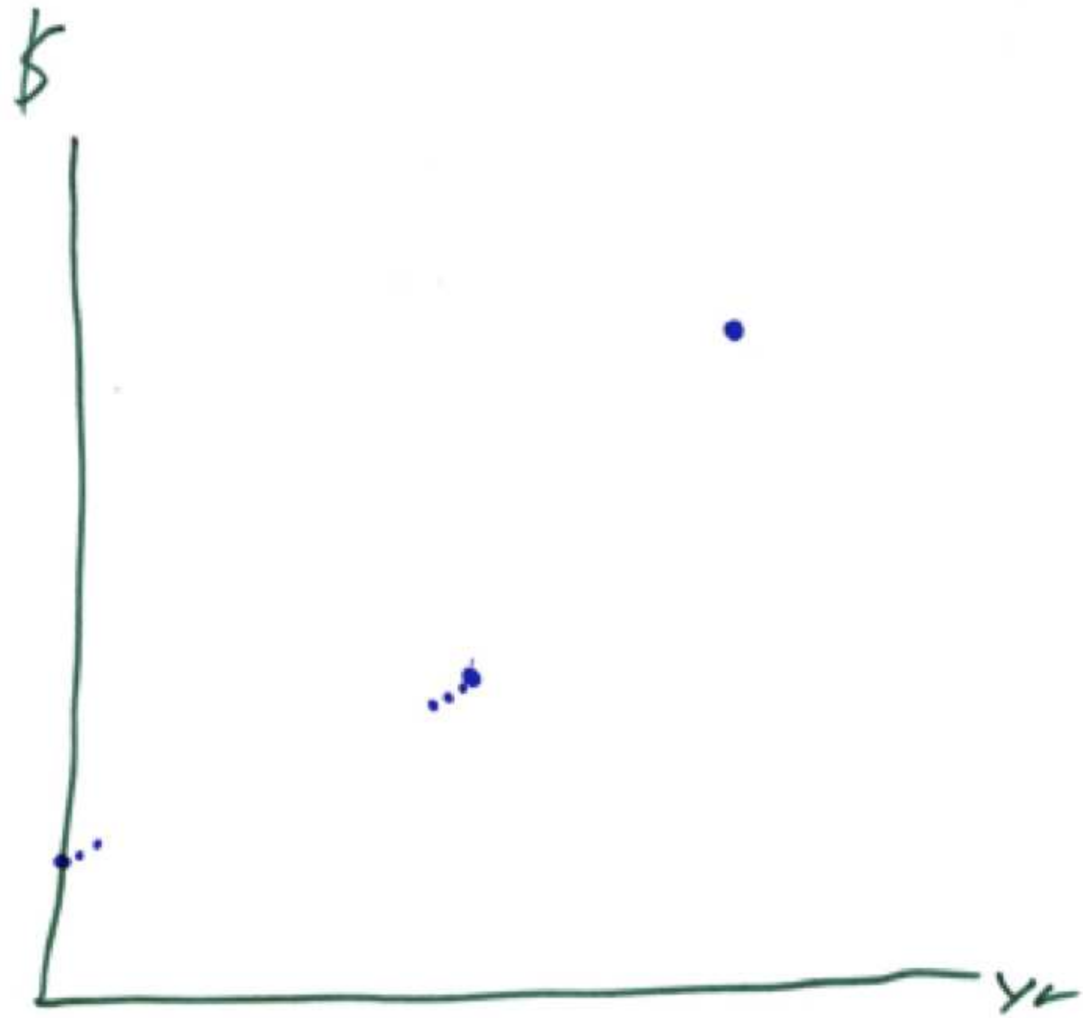

Figure 7. Tiffany fills in the first second interval with two sequences of points.

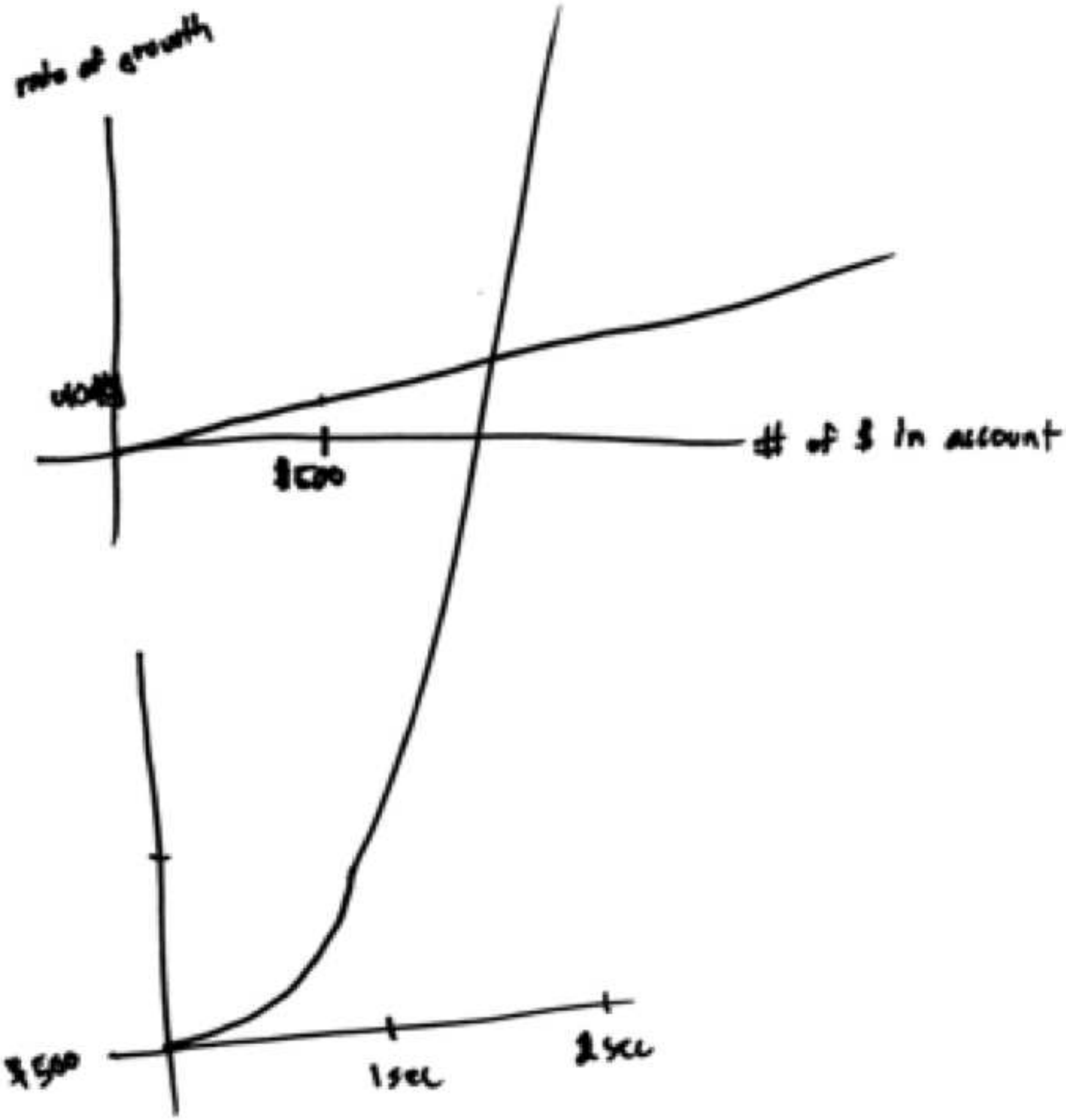

Figure 8. Derek's phase plane graph (top), and his corresponding graph of the account value over the first two seconds of the investment (bottom).

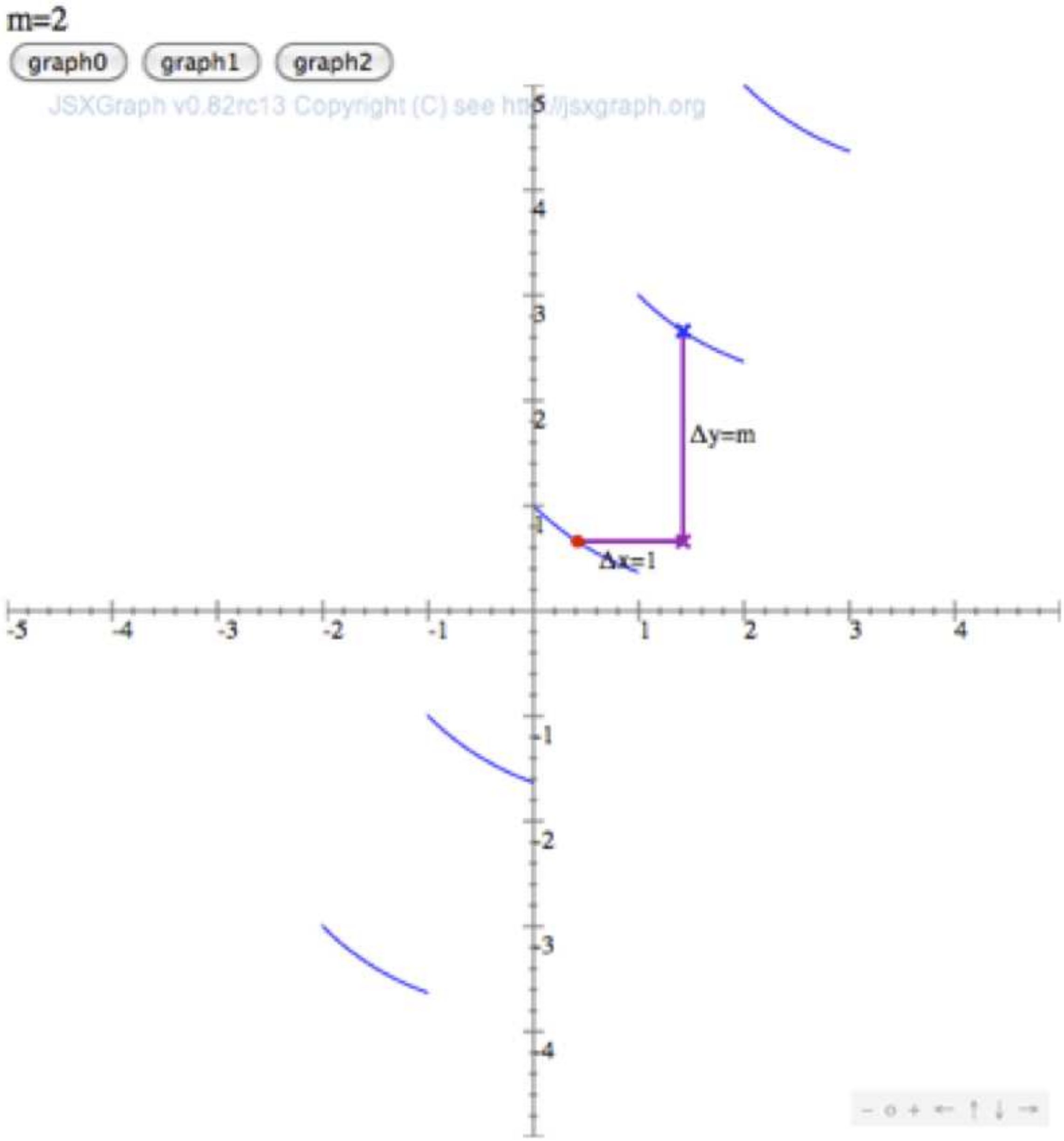

Figure 9. "Over 1 and up 2" fits a graph that has negative slope (almost) everywhere.