Chunky and Smooth Images of Change

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Abstract

Characterizing how quantities change (or vary) in tandem has been an important historical focus in mathematics that extends into the current teaching of mathematics. Thus, how students conceptualize quantities that change in tandem becomes critical to their mathematical development. In this paper, we propose two images of change: chunky and smooth. We argue that these images of change have different mathematical roots. Chunky images of change are based in countable and completed amounts, whereas smooth images of change are based in imagining a continually changing experience. Using empirical examples of student work, we illustrate these images of change and their implications.

Keywords: covariation; variation; algebra; trigonometry; continuity;

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Historically, imagery has played a prominent role in the study of change (Clagett, 1968, Edwards, 1979). The work of a fourteenth century scholar, Nicole Oresme, drew on images of change as a continuous, measurable quantity: “Everything measurable, Oresme wrote, is imaginable in the manner of continuous quantity” (Boyer, 1991, p. 264). Oresme examined change as an intensity of a measurable object, indicating a line to be a “fitting” representation of an intensity, because both intensities and lines could be increased or decreased without bound (Clagett, 1968). Such envisioning suggests a continuous image of change on the part of Oresme, and continuous images of change afforded Oresme’s distinguishing between variation in the intensity of change. Appealing to an imagined point moving along a line, Oresme indicated three distinctly different types of change: a particular intensity throughout, a “regularly” increasing or decreasing intensity, and an “irregular” motion that could not be accounted for by regular motion (Clagett, 1968). The different intensities distinguish between what would later be referred to as constant and varying rates of change, suggesting at least two types of varying rates of change. Oresme’s continuous images of change were instrumental in advancing the study of change, hinting to later uses of graphical representations of varying quantities (Edwards, 1979).

Given the role images of change have played in the history of mathematics, investigating students’ images of change seems warranted. We argue that considering students’ images of change provides insight into how students conceive of variation and make sense of situations involving varying quantities. This paper illustrates affordances and constraints of two images of change—chunky and smooth—articulating the implications that different images of change might have on individuals' conceptions of variation and of situations involving varying quantities.

An individual's conceptualizing of variation involves creating images of change. By *conceptualizing*, we mean engaging in mental operation (Piaget, 1970). Operation indicates a mental activity that an individual is capable of executing without necessarily engaging in observable action. For example, an individual can envision how the volume and height of water might vary when a bottle is being filled without physically filling the bottle. By *images*, we mean more than just products of conceptualizations. Images and mental operations are reflexively related such that images constrain mental operations, and mental operations shape images (Thompson, 1994a).

By considering students' images of change in situations involving variation, several researchers have made progress in investigating students' consideration of important mathematical concepts such as rate of change/differentiation, limit, and accumulation/integration (e.g., Carlson et al., 2002; Kidron, 2011; Oehrtman et al., 2008; Rasmussen, 2001; Thompson, 1994a; Thompson, 1994b; Zandieh, 2000). Collectively, this research suggests that the mechanics of how students conceptualize variation influences the mathematics that they construct. In this paper, we make distinctions between students' images of change that appear to involve different mental operations. Specifically, we posit two images of change that have emerged from our work with secondary (Castillo-Garsow, 2010; Johnson, 2012) and university students (Moore, 2012).

To illustrate distinctions between different images of change, we first use the bottle problem, originally developed by the University of Nottingham's Shell Centre

(Swan & the Shell Centre Team, 1999). Imagine a bottle being filled with liquid. Now consider how one might conceive of the volume of the filling liquid as changing relative to the filling liquid height. One way is to envision sections along the height of the bottle and determine/estimate amounts of volume in each section. Images generated from this way of thinking involve corresponding amounts of volume and height. Alternatively, one might envision both the volume of the liquid and the height of the liquid as changing together so that each is *continually* increasing. Images generated from this way of thinking involve volume and height continuously taking on different amounts as the filling progresses. The former way of thinking could be likened to filling a bottle with successive cups of liquid. As such, changes in volume and height occur in discrete chunks. The latter way of thinking could be likened to filling the bottle from a hose. As such, changes in volume and height continually progress. Not only do these different ways of thinking offer contrasting perspectives on the bottle problem, they also indicate how students might draw on differing images of change when constructing relationships between changing quantities (in this case, the volume and height of water in a bottle).

A Task Situation and Two Students' Images of Change

As suggested by the bottle problem example, students working on the same task can draw on different images of change to make sense of varying quantities. To demonstrate, we provide an example of a task from a teaching experiment (Steffe & Thompson, 2000) conducted by Castillo-Garsow (2010). Prior to the task, two Algebra II students, "Derek" and "Tiffany," worked tasks involving a variety of bank account policies. For each policy, the students were to develop graphs and equations that described the behavior of an investment under the policy, leading to the guided reinvention of a simple interest formula, compound interest formula, and differential equation for continuous compounding. The example discussed here is from the differential equation portion, where the students are developing an understanding of what the equation they created might indicate about the behavior of an investment over time.

The Task Situation

The policy was posed as follows:

> The Savings Company (SayCo) also competes with Yoi Trust and Jodan. SayCo's PD8 account policy is as follows: if you have one dollar in your bank account, you earn interest at a rate of 8 cents per year. For each additional dollar, your interest increases by another 8 cents per year. If you have fractions of a dollar in your account, your interest increases by the same fraction, so 50 cents earns interest at 4 cents per year. Here is SayCo's new feature: at any moment you earn interest, SayCo adds it to your account balance; every time your account balance changes, SayCo pays interest on the new balance and calculates a new growth rate. Why is SayCo's PD8 the most popular account?

First, the students were asked to write a rule for a function predicting the rate of growth of the account from the value of the account. Later, the students created a graph of this equation. Finally the students were to use the graph they had created (relating rate of growth of the account in dollars per year and the value of the account in dollars) to create a graph relating the value of the account in dollars in relation to time. Data that follows come from the final task, in which the students were required to use the graph of rate and amount (the phase plane graph) to create a graph of amount and time.

Tiffany

After sketching a phase plane graph of a financial policy where the dollar per-year rate was proportional to amount (Figure 1), Tiffany needed to create a graph of the account value over time. Essentially, Tiffany was tasked with determining a graphical solution to the differential equation $dy/dt=.08y$.

[Figure 1 about here]

While working the task, Tiffany imagined time passing in equal-sized chunks based on a standard time unit (years, months, days, hours, seconds, tenths of seconds, etc.). She reasoned that as time changed in equal-sized chunks, each successive change in the account value would be larger than the one before. Initially, Castillo-Garsow asked Tiffany about chunk sizes of one second, but then attempted to direct Tiffany to chunk sizes of smaller than one second:

1	Carlos:	Now tell me what happened during that second.
2	Tiffany:	That second during that second you earned a little bit of money so now they're gonna recalculate and stuff for your new-
3	Carlos:	OK,
4	Tiffany:	-rate.

Tiffany imagined "what happened during that second" as being someone doing the calculations needed to find the value of the account at the completion of that second and then determining the rate for the next second. However, Castillo-Garsow intended this question to be about "what happened" *to the account within* that second, a perspective that he had been attempting to help Tiffany develop by asking her about the values of accounts at successively smaller intervals of time. Castillo-Garsow subsequently drew Tiffany's attention to the account value at tenth and thousandth of a second intervals, and he then asked her to create a graph of the function for the first 3 seconds. Tiffany chose a chunk size of 1 second, and scaled the graph so that the points were clearly distinguished.

[Figure 2 about here]

The contrast between Tiffany's continuous line graph in Figure 1 and the point graph she drew in Figure 2 can be explained by the different thinking used to create each graph. Tiffany created the line graph in Figure 1 from an equation that she recognized as the equation of a line, and drew the shape that she associated with that equation (Castillo-Garsow, 2010). While creating Figure 2, however, Tiffany did not have access to an equation or know that a graph would be exponential. Without that information, Tiffany based her solution (Figure 2) on her understanding of the situation: that each successive value must be calculated (a process that takes some non-zero amount of time), and that after each (imagined, not necessarily carried out) calculation, the result would be an account value that changed by more than the previous change in account value. This reasoning led her to construct a graph consisting of separate points, a separation that was important to Tiffany. In fact, her graph shown in Figure 2 was very large, taking up nearly an entire 8.5 x 11 piece of paper, and on that scale the points that she graphed were very far apart. Although the horizontal axis was scaled in years, Tiffany did not place the three points close together, near the origin, but rather emphasized the distinctiveness of each point by spreading them across the page. Reflecting on Tiffany's actions, choosing a 1 second chunk size is natural from the point of view of a question about "3 seconds." However there are multiple ways of interpreting the task of graphing over 3 seconds.

Derek

Another meaning for the charge of "graphing over 3 seconds" can be found in Derek's solution to the same task. Whereas Tiffany imagined change in terms of one-second chunk sizes, Derek imagined change as occurring *smoothly*:

1	Derek:	What I don't get is like it does it if you have the dollar and then you just put in a dollar at any point does it change right away?
2	Carlos:	Umm well it seems like the way this was written that if you put in another dollar it would change right away. But let's imagine for a moment, umm, that we're only putting money in this account one time. We're only investing in this account one time.
3	Derek:	So does the eight cents interest also affect it? The change of growth rate?
4	Carlos:	Umm I'm not sure what you mean by that.
5	Derek:	Like if like if you have a dollar you put in a dollar and the growth rate changes right away? Well if you're getting money constantly is the growth rate increasing constantly?

Following his explanation, Derek sketched a graph in two chunks, completing a section for each second. Because his chunks resulted from an image of smooth variation (e.g., “changes right away” and “getting money constantly…growth rate increasing constantly"), rather than beginning their life as chunks, he produced a graph that is a continuous curve (Figure 3 bottom), which is drastically different than Tiffany's graph.

[Figure 3 about here]

Looking Across Tiffany and Derek's Work

Although Tiffany and Derek drew relatively similar graphs of the account in the phase plane, their solutions to the phase plane task ended very differently. This outcome can be explained by their differing images of change. When creating the graph, Tiffany conceived the account value as changing *after* a specified interval of time. In contrast, Derek conceived the account value as continuously changing with continuous changes of time. Just as a student can imagine volume and height accumulating in a bottle in different ways (e.g., filling cup by cup or with a hose), the work of Tiffany and Derek, respectively, suggest two distinctly different images of change: *chunky* and *smooth*.

Chunky Images of Variation

Conceptualizing chunky variation involves imagining change as occurring in completed chunks (Castillo-Garsow, 2010; Castillo-Garsow, 2012). Ongoing change is generated by a sequence of equal-sized chunks, and this makes measuring change essentially about counting how many chunks have occurred. Chunky variation is characterized by two features: a unit chunk whose repetition makes up the variation, and the lack of an image of variation within the unit chunk. A key aspect of the chunk is that it is conceived of atomically; the chunky thinker imagines change as occurring in amounts equal to the size of the chunk. In doing so, the chunky thinker attends to the edges that define the chunk and the number of chunks. The intermediate values within the chunk "exist," in the sense that they are needed to fill out the chunk, but they receive little or no attention. For example, even when Castillo-Garsow asked Tiffany to talk about what happened *during* a second, she interpreted the question in terms of the account value *at particular instances of seconds*; nothing of importance happens within the chunk because the entire chunk is imagined all at once.

Chunky Thinking and the Existence of Holes

Because a chunky thinker works in atomic units, she reasons in discrete points using the resolution of that unit. We refer to the spaces between these discrete points as *holes.* By holes, we do not mean that nothing exists in the spaces, but instead emphasize that a key factor in chunky thinking is an essentially singular focus on discrete points that exist at the edges of the chunks. Both the size of the chunk and the scale of a representation can illustrate the inherent nature of holes in chunky thinking. In Tiffany's case, instead of continuing with a chunk size of a thousandth of a second or creating a graph on a scale of years (both of which would obscure the holes), she chose a chunk size and scale that made the holes explicit. No matter how small she cut up her chunks, Tiffany always conceptualized them as chunks, and imagined them at a scale where she could see the chunkiness. Holes played a prominent role in Tiffany's thinking, even at times when she was less aware of the holes (Castillo-Garsow, 2010).

The place of holes in Tiffany's thinking also points to *where* roots of change are for a chunky thinker. When Tiffany imagined finding the value of the account at one tenth of 1 second, she imagined cutting up a completed 1 second chunk into ten tenth-of-a-second chunks. In Tiffany's experience, the event "finding the value of the account at 1 second" happened *before* the event "finding the value of the account at one tenth of 1 second," because the value at 1 second was needed to calculate the value at one tenth of 1 second. This resulted in Tiffany's asynchronous experience of variation. Tiffany imagined what happened at one second *before* she imagined what happened at one-tenth of 1 second. More generally, it was necessary that Tiffany imagined what happened at the edges of a chunk *before* she could consider what happened *during* the chunk. Chunks (and their holes) come first, and values within those chunks are conceived of after (if at all). When time is imagined in terms of chunks, a mismatch exists between a temporal ordering of values within a situation and an experience of those values by a student doing the imagining. This mismatch makes it seemingly impossible for a chunky thinking student to imagine a situation dynamically while simultaneously imagining the mathematics of it.

The Dependence of Chunk Size on External Features

When an independent variable varies in chunks, the dependent variable can be calculated from the independent variable. Such an approach is commonly used by students graphing functions in U.S. schools: students plot a few points in order to get a sense of the function and then connect the points. This method of graphing and the instruction that supports it was criticized in detail in Leinhardt, Zaslavsky, & Stein's (1990) review of the literature on graphing. We argue that this type of graphing is not *covariational* (Carlson et al., 2002; Confrey & Smith, 1994; Saldanha & Thompson, 1998). Rather, it is variational, as plotting points only requires imagining x varying, or said more strongly, only requires imagining different x values. Because the space between each point remains unevaluated, this type of graphing supports a chunky conceptualization of variation. Even if a student returns to a chunk and plots additional points within the chunk, then the student is still engaging in chunky thinking on a smaller scale. Although a single variation point plotting approach enables students to graph a multitude of functions, one issue that arises with this single variation approach is that the chunk size needs to be determined prior to graphing–often by a secondary understanding of the problem.

An example of this issue emerges in the work of "Bob," an undergraduate student enrolled in a secondary mathematics content for teaching course (Moore, 2012). Bob was prompted to draw a graph of $y=\sin(3x)$. He first gained a sense for the function by plotting points for x values of 0, $\pi/2$, and π. After plotting these points, Bob seemed confident enough in his sense of the function to say: "I guess it just reflects [$\sin(x)$]," and drew the graph shown in Figure 4. Beginning his graphing process by plotting in equal sized chunks, Bob chose to use a chunk size of $\pi/2$, which would have been an appropriate chunk size were he graphing $y=\sin(x)$ (the function graphed during a previous teaching session). Because the chunk size that he chose aligned with critical points of $y=\sin(3x)$, the graph appeared to him to be a trigonometric curve with period 2π rather than period $2\pi/3$. After creating the graph, Moore attempted to draw Bob's attention to values within his chunk by asking him for the value at $x=1$. Bob quickly concluded that the value would be $\sin(3)$ and added a point for the value of $\sin(3)$ based on the graph he had drawn (the reflection of $y=\sin(x)$). Because—to a chunky thinker—the important mathematics occurs at the edges of the chunk, and because Bob's initial chunk size was too large for the situation, plotting sin(3) as negative was not problematic to Bob.

[Figure 4 about here]

Although reducing the size of the chunk is sufficient to address Bob's specific situation, no chunk size is sufficient to cover all situations dealing with trigonometric functions. If Bob had adopted a standard of selecting a chunk size of π/6 instead of π/2, he would have correctly graphed the function $y=\sin(3x)$. However, to graph $y=\sin(9x)$, a smaller chunk size is needed. Although a student using chunky thinking might be able to reduce the size of a chunk, the student still would be imagining the change as occurring in chunks (e.g., Tiffany's use of 1 second chunks, then tenth of a second chunks). Therefore, reducing the chunk size is not sufficient because no matter how small the chunk size, the focus remains on discrete points. The consequences of this focus on discrete points are well documented, with Leinhardt, Zaslavsky and Stein (1990, p. 45) identifying a "pointwise focus" as one of the three broad areas of learning problems that students have when graphing – a problem that is characteristic of chunky thinking. In order to move beyond a pointwise focus, some other way of thinking seems necessary to make sense of change that might be occurring within a chunk and to guide the selection of chunk size.

Smooth Images of Variation

Conceptualizing smooth variation involves imagining a change in progress (Castillo-Garsow, 2010; Castillo-Garsow, 2012). Ongoing change is generated by conceptualizing a variable as continuously taking on different values in the flow of time. A smooth variable is always in flux. The change has a beginning point, but no end point. As soon as an endpoint is reached, the change is no longer in progress.

Smooth variation is not the same as chunky variation cut up really small; it is an entirely different conceptualization of variation. The difference may be best illustrated with a metaphor: imagine watching a movie. Smooth action appears to occur on the screen. Behind the scenes, however, a movie is a series of still images played at 24 frames per second. Imagine flipping through the reel of film by hand, looking over each image on a light box. There is no sense of motion, and the character depicted on film appears at a fixed location in each frame. This experience is completely different than watching the movie; watching the movie is looking at change in terms of a smooth progression. Although a movie is chunky, one does not perceive a movie as chunky because it is changing faster than one can conceive the chunks. The smooth effect is entirely perceptual. If one were to slow the film down one could see discontinuities. Similarly, the difference between chunky variation and smooth variation is entirely conceptual: does one imagine the frames (chunky), or does one imagine the motion (smooth)?

Smooth variation by its nature does not provide numerical solutions to problems. Returning to Derek's graph (Figure 3 bottom), one might notice that although Derek described the behavior of the function, there are no numbers marked on the graph that would be useful for estimating values of the account. Taking time to calculate a value would stop change in progress, unless those calculations were made sufficiently quickly (with the aid of technology) so that the chunkiness is present but hidden. This is because smooth variation involves imagining a change that is always in progress, and thus it entails a capacity to reason without specified numbers and calculations.

Using Smooth Thinking to Consider Chunks

Although smooth thinking and chunky thinking are fundamentally different, they are not unrelated. We find it difficult to conceptualize how chunky thinking might be a root for smooth thinking (e.g., no matter how small the chunks, chunks are always present). However, we argue that if smooth thinking were a root for chunky thinking, it might form a foundation for a powerful form of chunky reasoning.

Consider Hannah, a tenth grade student, who responded to an adaptation of the bottle problem described earlier in this paper. The adaptation, developed by Johnson (2010), required Hannah to sketch a viable bottle shape that could be represented by a nonlinear graph representing the volume of liquid as a function of the height of liquid in a filling bottle. When drawing a viable bottle shape Hannah considered different sections of the bottle, imagining the height as increasing continuously. Unlike chunky thinkers, who might base their sections on particular amounts of volume and height, Hannah based her sections on when the volume of filling liquid would increase at different intensities with respect to the increasing height (Johnson, 2012). Hannah's description, along with her gestures, indicated her imagining a change in progress, suggesting that she was using smooth thinking to make sense of how the height and volume of liquid were covarying. Although Hannah's sectioning the bottle created chunks, the chunks came as a consequence of smooth thinking and noting varying intensities of change; Hannah's chunks did not entail holes as variation within the chunks was at the forefront of her conception of the situation.

Previous research has articulated the construct of continuous covariation (Saldanha & Thompson, 1998; Thompson, 2008; Thompson, 2011), a recursive process in which a variable is conceived of as varying smoothly over a chunk. Within a chunk, smooth variation takes place, variation that can be cut up into smaller chunks such that each chunk is imagined as undergoing smooth variation sequentially. The smooth variation within these subchunks also could be conceived of as occurring in smaller chunks, and so on (Castillo-Garsow, 2010; Castillo-Garsow, 2012; Saldanha & Thompson, 1998; Thompson, 2008; Thompson, 2011). Even though chunks play a prominent role in continuous covariation, Hannah's example shows chunks do not entail holes if they are created from smooth variation.

An example of continuous covariation emerges in the work of "Zac," an undergraduate precalculus student interviewed by Moore (2010, 2012). Zac was prompted to draw a graph of the distance of a passenger from the ground in relation to the time elapsed on a Ferris wheel ride (for a wheel radius of 36 feet):

1	Zac:	Ok. So a really easy way to do this is divide it up into four quadrants (*divides the circle into four quadrants using a vertical and horizontal diameter, Figure 5*). 'Cause were here (*pointing to starting position*), for every unit the total distance goes (*tracing successive equal arc lengths*), the vertical distance is increasing at an increasing rate (*writing*

i.i.)...Then, uh, once she hits thirty-six feet, halfway up, it's still increasing but at a decreasing rate (*tracing successive equal arc lengths, writing i.d.*)...Uh, then when she hits the top, at seventy-two, it's decreasing at an increasing rate (*tracing successive equal arc lengths, writing d.i.*)...And then when she hits thirty-six feet again it's still decreasing (*making one long trace along the arc length*), but at a decreasing rate (*tracing successive equal arc lengths*, *writing d.d.*)

[Figure 5 about here]

At first, Zac's actions, which involve cutting up the Ferris wheel into quadrants (e.g., chunks) and tracing successive chunky arc lengths (e.g., subchunks), suggest chunky thinking. However his motions in tracing out the arc lengths and his terminology (increasing, increasing rate, and the use of the present tense for variation) seem to indicate the imagining of change in progress, a hallmark of smooth thinking. To Zac, his chunks and subchunks contained covarying quantities. Further, like Hannah, the imagining of change in progress seemed to support Zac's consideration of variation in the intensity of the rate.

Mathematical Differences in Chunky versus Smooth

A key distinction between smooth and chunky thinking lies in the nature of the product generated as a result of the thinking. Simply put, chunky thinking generates chunky conceptions of variation, whereas smooth thinking generates smooth conceptions of variation, with these conceptions producing different mathematics. Although a smooth conception of variation involves imagining a changing number or magnitude, it also involves attending to all states continuously without privileging any sort of unit value that would form the basis of counting. Therefore a smooth conception of variation does not immediately generate products that could be counted. In contrast products of a chunky conception of variation are always countable, because no matter how small the chunk, discrete points always exist and can be counted.

Historically, it was smooth images of change that supported Oresme's consideration of variation in the intensity of change. The work of Oresme in the fourteenth century influenced the development of symbolic representations of rate of change that emerged in the seventeenth century (Edwards, 1979). Only after representations drawing on smooth images of change emerged did representations drawing on chunky images of change develop, and it was only with the advent of modern analysis that fully a chunky (epsilon-delta) calculus was possible. Since we ask students to reason about change and rate long before they take analysis, it seems reasonable that beginning with smooth images of change could create foundations for students' consideration of the difficult to learn mathematics concepts of limit, rate of change/differentiation, and accumulation/integration.

Concluding Remarks

We argue that smooth thinking is a more powerful root for students' images of change than chunky thinking, in part because of the mathematical difficulties that chunky thinkers such as Tiffany and Bob seem to have that smooth thinkers such as Derek, Hannah and Zac do not (Castillo-Garsow, 2010, 2012; Johnson, 2012; Moore, 2010, 2012); but also for philosophical reasons. Because it seems impossible for a smooth thinker to experience imagining an ongoing change forever, we conjecture that at some point the imagining of ongoing change must stop, at which point a smooth thinker would have imagined a completed change (a chunk). Therefore, smooth thinking would entail a capacity to think in chunks (or, at least at its foundation, one chunk). In contrast, in our experience with students, chunky thinking does not seem to entail a capacity to think smoothly. Future research could explore how students develop smooth and chunky thinking. Investigating the consequences of smooth and chunky thinking across various mathematical topics might also provide elaboration on characterizations of smooth and chunky images of variation and their role in mathematical thinking and learning.

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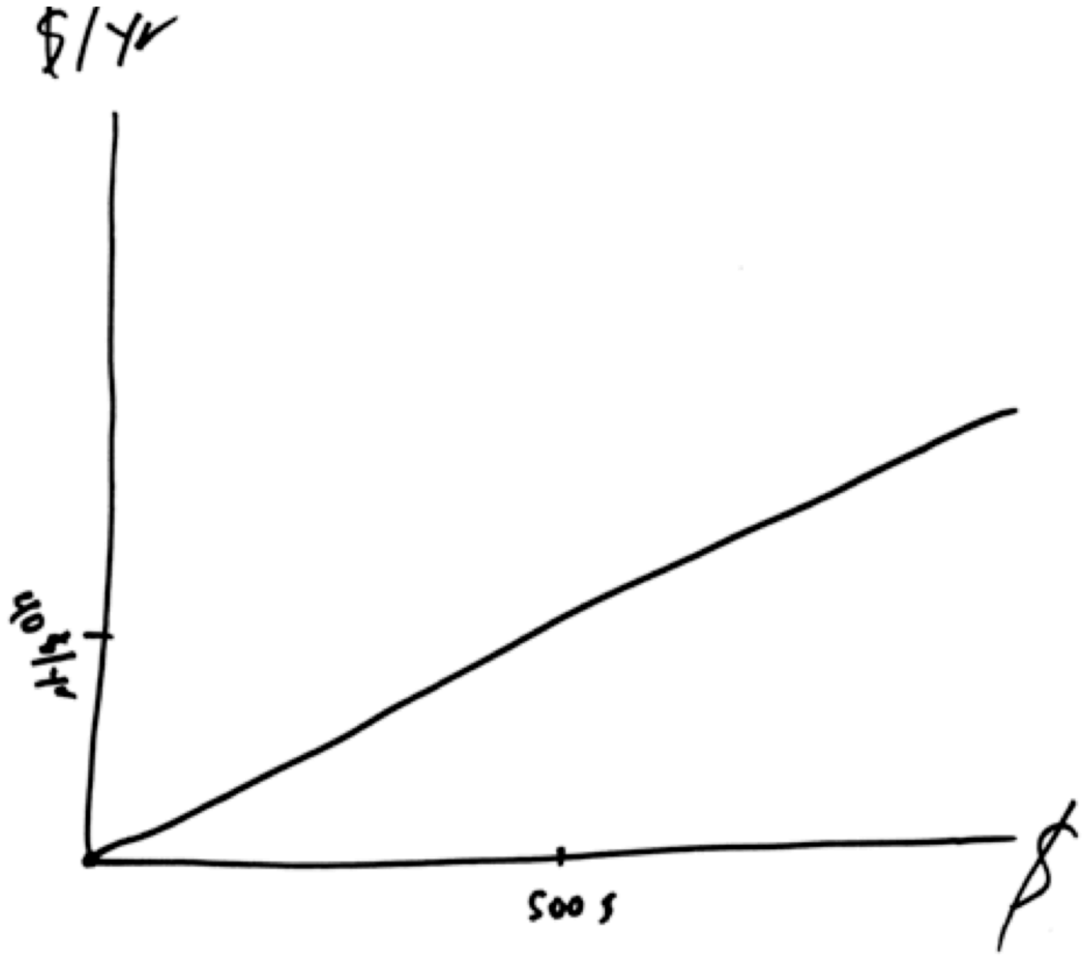

Figure 1. Tiffany's graph showing the proportional relationship between the dollar per year growth rate of an account and the value of the account.

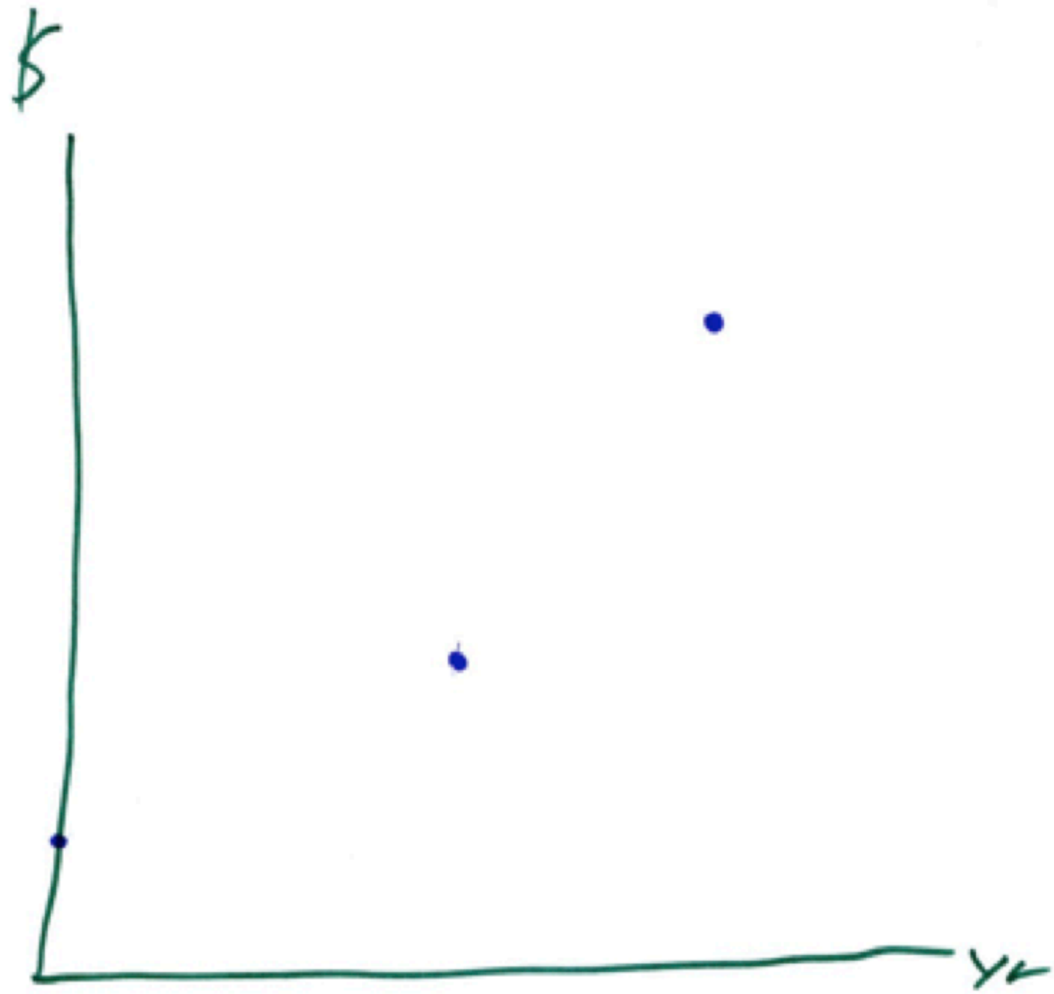

Figure 2. Tiffany's solution to the phase plane problem. The 3 points marked are each one second apart while the horizontal axis reads "yr" for years, from Castillo-Garsow (2010).

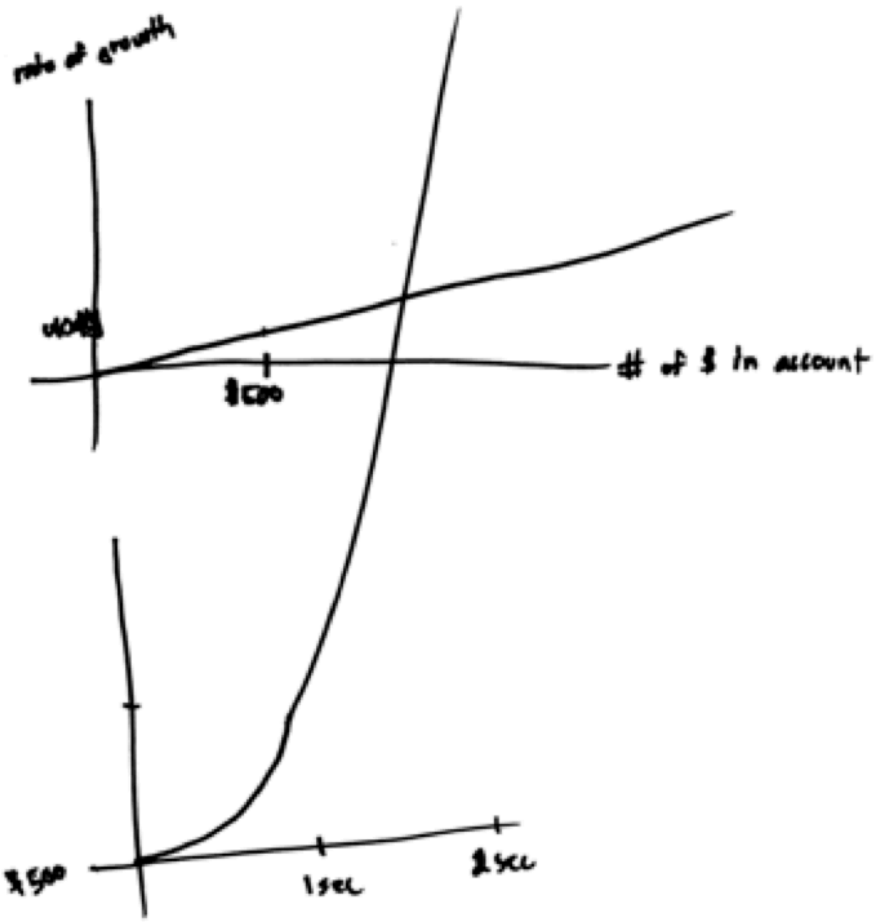

Figure 3. Derek's graph of the account in the phase plane (top) and his solution to the phase plane problem (bottom), from Castillo-Garsow (2010).

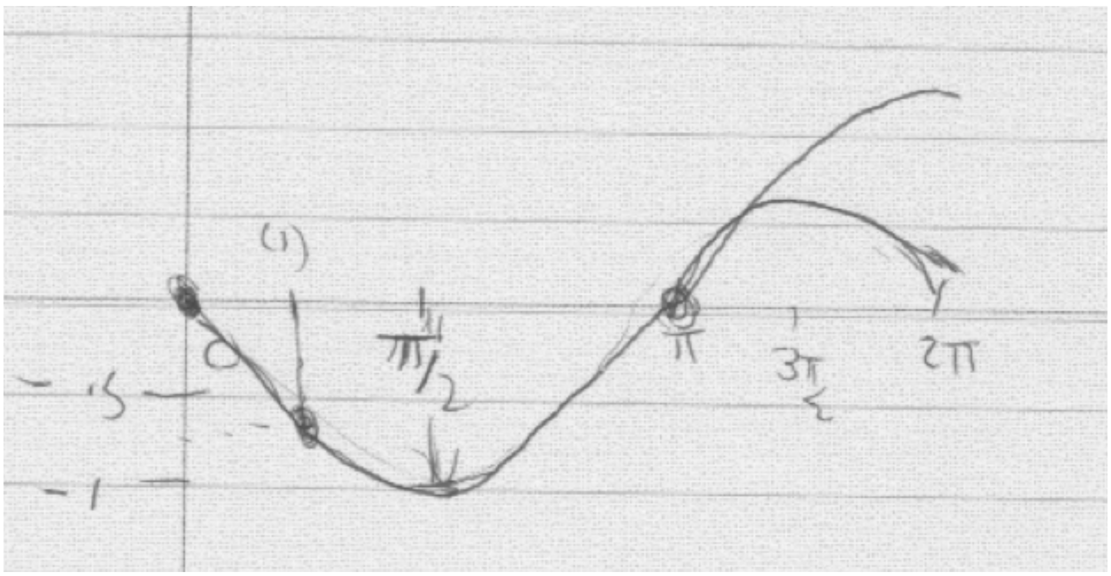

Figure 4. Bob's graph of y=sin(3x), from Moore (2012).

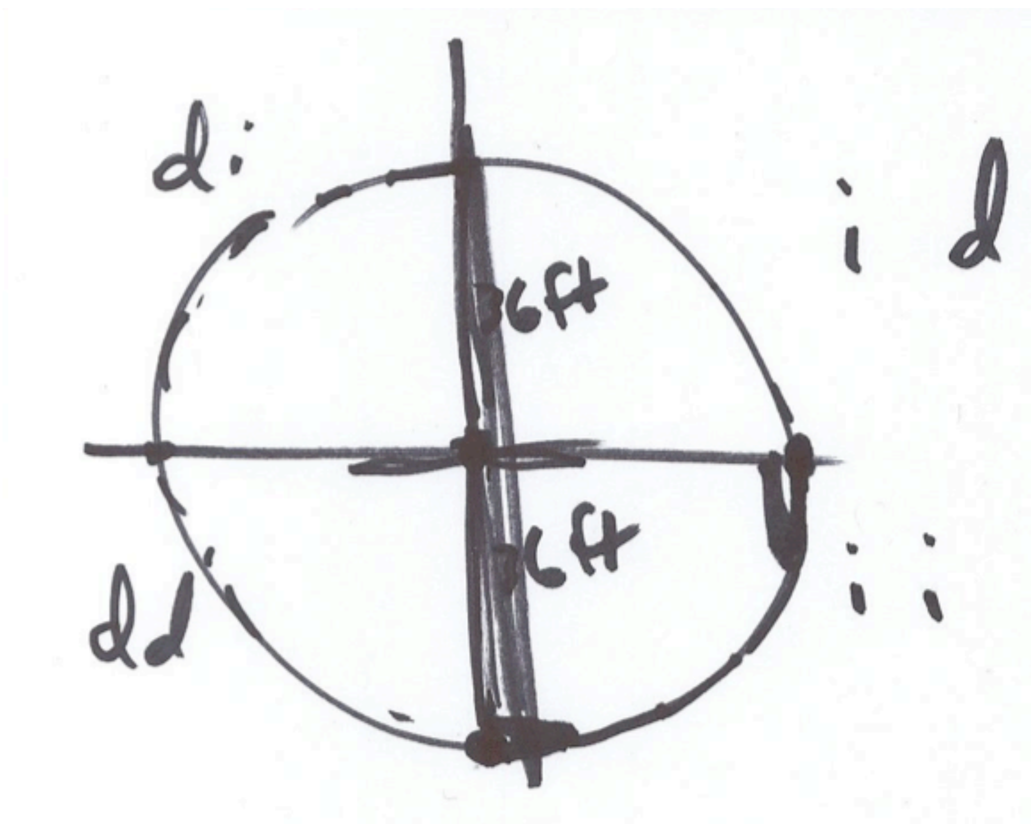

Figure 5. Zac's diagram of the Ferris wheel, from (Moore, 2012).