Why REUs matter

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1 History and Introduction

The Cornell-SACNAS Mathematical Sciences Summer Institute (CSMSSI) was a mathematical biology research experience for undergraduates. REU founded in 1996. One year later, the program was renamed as the Mathematical and Theoretical Biology Institute (MTBI). In 2004, MTBI moved to Arizona State University (ASU), where it merged with a K-12 program, the Institute for Strengthening the Understanding of Mathematics and Science (SUMS). In 2008, MTBI/SUMS expanded again to become embedded in what is now the Simon A. Levin Mathematical, Computational and Modeling Sciences Center (MCMSC). The purpose of the larger center is now to connect MTBI's education-through-research mission directly to undergraduate and graduate programs in the mathematical sciences and to cutting edge research activities at the interface of the mathematical, life, and social sciences.

Now, in our 20th year, MTBI/SUMS has been extraordinarily successful by most traditional measures. From 1996 through its 2014 summer program, MTBI has recruited and enrolled a total of 423 regular first-time undergraduate students and 91 advanced (returning) students. MTBI students have been prolific researchers, with 180 technical reports [1], and a large number of refereed publications. (including but not limited to the following representative publications, [17, 43, 14, 16, 15, 19, 24, 25, 26, 27, 28, 31, 33, 36, 37, 40, 46]).

[1] http://mtbi.asu.edu/research/archive

MTBI has also been successful in recruiting and retaining students in mathematical sciences. Through December 2014, 249 out of 357 (70%) of U.S. MTBI student participants had enrolled in graduate or professional school programs and 109 MTBI student participants have completed their Ph.Ds. 74 students have received their Ph.D.s since 2008, for a Ph.D. completion rate of a little over 10 Ph.D.s per year.

MTBI has received funding from Cornell University, Arizona State University, Los Alamos National Laboratory, the Sloan Foundation, the NSA, and NSF. NSF and NSA currently fund MTBI each at roughly 120K per year, with graduate mentors supported by teaching and research assistantships (127 since 1996) mostly funded via university funds, and Sloan fellowships.

The program has also received external recognition in the form of multiple national awards. The Director of MTBI was awarded a Presidential Award for Excellence in Science, Mathematics and Engineering Mentoring (PAESMEM) in 1997 and the American Association for the Advancement of Science Mentor Award in 2007. Also in 2007, the AMS recgonized MTBI as a Mathematics Program that Makes a Difference. MTBI's partner high-school program, SUMS, was recognized the Presidential Award for Excellence in Science, Mathematics, and Engineering Mentoring in 2002, and MTBI received the same award in 2011.

The reason for MTBI's extraordinary success is that the primary goal of MTBI has always been to effect social change. MTBI/SUMS's mission is to encourage student — especially women, Chicano/Latino, Native American, and African American students — to pursue advanced degrees in the computational and mathematical sciences, with particular emphasis in the applications of mathematics to the life and social sciences. 249 (70%) of MTBI undergraduates are underrepresented minorities and/or members of the Sloan Pipeline Program (Underrepresented minorities (URMs) include Hispanic, African-American and Native American students). Since January 1995, 69 (77%) of US citizen or permanent resident alumni completing PH.D.s have been URMs, and 51% of all MTBI Ph.D. recipients are women. Research, funding, and external recognition are means to purse the ultimate mission of social change, and the mission of social change results in quality research, funding, and recognition. Students — particularly under-represented students from impoverished backgrounds — have a keen interest in social change. Faculty are inspired by this mission as well many guest faculty participate frequently, voluntarily and without pay.

In short, we have built and continue to build a community model to start new undergrad majors and PhD degrees over the past 19 years — its effectiveness is tied to the spirit of community, volunteerism, and service that pervades every aspect of the program. From this perspective of effecting social change through mathematical research, MTBI has succeeded, but there remains much to do. It is this ongoing mission and the continued sense of dissatisfaction with MTBI's successes to date that has driven MTBI faculty and staff to continue to do better, to continue to grow over the past 19 years from a simple summer REU to a national recognized pipeline program that tracks and supports under-represented students from high-school to tenure.

In particular, ever since the merger with SUMS in 2004, MTBI/SUMS has become increasingly interested in K-12 education. The addition of SUMS and partnership with other high-school programs in Arizona has improved our ability to serve the URM population by giving us the ability to grow URM talent for MTBI, rather than rely solely on the recruitment of students who make it to college on their own — similar to the way that the REU grows talent to be recruited to graduate school.

However this partnership has also made us aware of other ways in which K-12 partnerships can be mutualistic. A short example of this type of collaboration is the Center's newly funded initiative to reintroduce a teaching and mentorship component to our programs. MTBI will be collaborating with Dr. Raquell Holmes of ImproviScience to provide educational research-based workshops on mentoring and collaboration for both student mentors and faculty.

More generally, we are proposing two more initiatives to strengthen mutually-beneficial ties between mathematical science REUs and communities of educators. We would like other REUs to be inspired by the goal of social change and join us in these or similar initiatives. We intend to show how REUs are uniquely positioned to improve effect social change by collaborating with mathematics education researchers and by recruiting from the population of qualified teaching majors. We will do this using examples from mathematics education research as well as from MTBI/SUMS itself; and in the process we hope to give the reader a more detailed picture of how MTBI works, the mechanics of its successes to date and, we hope, the desire to have a dialogue that help us to improve what we do by learning first-hand what other groups have done so well over several decades.

2 The Mechanics of MTBI/SUMS

There are a number of factors that play a role in the educational success of MTBI/SUMS. The most important seem to be the mathematical content, the reversal of hierarchy, and the community model.

MTBI/SUMS summer sessions are divided into two parts. For the first three and a half weeks, students attend lectures and do homework (*the mathematical content*). For MTBI this content includes difference and differential equations, statistics, stochastic processes, agent based modeling[2], and computer simulation. The program roughly follows the text of Brauer and Castillo Chavez [4], supplemented by guest lectures. For SUMS, this content is typically difference equation modeling only, based on an idea from Robert May [30].

The second half of the program is student-driven group research projects. In MTBI's initial prototype year (1996), the projects were assigned as tasks. However, this did not generate the desired student buy-in or the experience of doing authentic research. In subsequent years, students were expected to design their own projects, while returning students, graduate students and faculty served in advising roles. Because students choose their topic of study, they frequently know more about the situation than the mentors (*flipped hierarchy*). The mentors contribute mathematical and modeling experience, but rarely situational knowledge [6, 7, 11].

Over the entire program, the faculty deliberately work to create a research community (*the community model*). MTBI is a residential program, and during the first half of the program, participants are deliberately given far more work than they can complete on their own, forcing them to make use of their peers. Even in groups, students typically work 10-12 hours per day, six days a week throughout the eight weeks. In order to explicitly encourage collaboration, students are organized and participate in weekly group meetings. These groups then form the foundation of the collaborative research project. On alternating weekends, students participated in organized activities such as day trips, water parks, a Fourth of July celebration and other social outings. These activities built esprit-de-corps among the students and facilitated their teamwork in research. Students are also encouraged to return as advanced students, graduate students, post-docs, and faculty – building

[2] models of large numbers of individuals acting according to pre-determined algorithms; usually, but not always simulated via computer

long-term bonds with the research community and serving as role models for new students.

The dynamics of the community model of MTBI have been discussed in detail elsewhere [17, 11]. So in this article, we will discuss the importance of the other two factors: the mathematical content and the reversal of hierarchy.

3 The reversal of hierarchy

Research modeling experiences form the keystone of the MTBI/SUMS summer experience. The entire program is designed to prepare students to successfully pursue a research modeling experience. Students are trained in mathematical biology techniques, but in the final half of the session, they apply these techniques to their own curiosities. In MTBI this has led to final projects in a variety of diverse topics including but not limited to: the three strikes law [42], gang recruitment [1], education [3, 20], immigration [13], political third party formation [38], mental illness [18, 21, 25], pollution [5],obesity [22], drug use [32, 43], and even MTBI itself[17]. Students choose these topics because they are personally meaningful. Nearly every one of these off-topic applications is chosen because a group member has personal experience with the problem. Either they themselves, or a family member, or a close friend has run afoul of a gang, or has dropped out, or struggles with a mental illness. This personal experience both motivates the group and supplies them with valuable insider expertise.

Because MTBI students choose their own research projects, they frequently know more about the topic than their graduate student and faculty mentors. This creates the "reversal of hierarchy" in which the students take the leadership role in the project, and the mentors serve as consultants. As consultants, the mentors provide missing mathematical expertise to judge the feasibility of the project, suggest appropriate tools, and tutor the students in any additional techniques they need for their project that were not covered in the general lectures.

SUMS students have similar experiences, but the mentor role is even more involved. Consider the most recent SUMS year, in which the students were trained in difference equation techniques for mathematical biology. Their interests, however, were diverse, and they did projects in economics, education, driving safety, and zombie epidemiology. In order to complete these projects, some student teams required additional mentoring in statistical and agent-based programming tools not originally covered by the difference equation curriculum.

This student leader with consulting mentor approach is a critical factor to the success of MTBI/SUMS. Without students choosing their own projects, a vital portion of the research experience is missing, and students are not truly doing research. Without the mentor in a consulting role, projects quickly exceed both the mathematical abilities of the students and the time they have available to complete the project. One of the key roles of the mentor is helping students slim their interests down to a project that can be completed in four weeks.

4 The importance of students doing research

Making the student the leader of the research is vital because it makes the mathematical experience real. When students are in the leadership role, they must draw on their own experiences to make sense of mathematics and to make decisions. Contrast this with experiences that students typically have in modeling in K-12 schools.

Reusser [35] describes the story of 97 first and second graders asked the following nonsensical questions:

> "There are 26 sheep and 10 goats on a ship. How old is the captain?"
> "There are 125 sheep and 5 dogs in a flock. How old is the shepherd?"

76 of those 96 first and second graders were able to provide a numerical solution to these questions, by finding an appropriate operation for the numbers that would result in an age. In the first task adding to 36 years old, and in the second task dividing to 25 years old.

The problem stems from the social norms that are quickly established in a mathematics classroom. Every question must have an answer, and that answer must be reached using the tools that are currently being studied. Or, as Reusser put it: "Almost every systematic dealing with ambiguity and unsolvability is factually excluded from textbooks, from curricula, and from the school setting where it even seems alien."[3]

[3]For more cases and anecdotes on lack of sense making in school mathematics, see Schoenfeld [41]

4.1 Pseudocontext

It should not surprise any of us that students learn that mathematics is not about making sense. Textbooks have long histories of including "modeling" exercises that do not require students to make sense, in some cases the tasks are actively hostile to making sense. Jo Boaler coined the term "pseudocontext" for refering to these types of tasks, saying:

> A restaurant charges \$2.50 for $\frac{1}{8}$ of a quiche. How much does the whole quiche cost?
> ...
> Everybody knows that people work together at a different pace than when they work alone, that food sold in bulk such as a whole quiche is usually sold at a different rate than individual slices, and that if extra people turn up at a party more pizza is ordered or people go without slices – but none of this matters in Mathland. [2, p. 52]

If students were required to make sense of the quiche task in the context of their real world experience of purchasing quiche, then the quiche task [2] [p. 52] is just as unsolvable as a captain problem from Reusser [35]. The quiche task is only "solvable" because of the social norms established in the classroom. The question is being asked in a math class, and a question in a math class must have a solution, therefore the normally incorrect assumption that the price of quiche is the sum of the price of its slices must be valid here. The same solvability assumption that allows students to answer the quiche task "correctly" is also what leads the majority of students to solve the captain task incorrectly.

4.2 Research modeling

Contrast this type of assigned task with the experiences of SUMS students, or with the following case from Resnick [34, p. 68–74]. Resnick describes two high school students, Ari and Fadhil, who were working with the agent based modeling program StarLogo at the same time they were enrolled in driver's education. Ari and Fadhil developed a curiosity: they wanted to know what caused traffic jams. Using StarLogo, Ari and Fadhil developed several simulations of drivers on a highway, and explored driver behaviors that contributed to or eliminated traffic jams. Although Ari and Fadhil did not succeed in controlling their simulated traffic jam, they discovered quite a bit about traffic jam behavior: that traffic jams moved as waves in the direction opposed to traffic, and that starting all cars at the same initial speed did not prevent a traffic jam, so long as the cars were unevenly spaced.

Ari and Fadil's story differs from the quiche example in two critical ways: first, there was no externally imposed task. Instead Ari and Fadil were pursuing their own curiosities. Secondly, Ari and Fadil were not looking for a solution to be validated by an authority figure. They were looking for the understanding that would satisfy their curiosity. Taken together, we have two high schools students pursing a non-mathematical curiosity with mathematical tools, while believe that no pre-existing solution existed. In other words, Ari and Fadil were engaged in mathematical research. This is exactly the sort of activity that quiche tasks do not prepare students for.

Based on the experiences of MTBI/SUMS students, as well as Ari and Fadhil's example, we propose three criteria for identifying research modeling activity:

1. The problem is based on the student's non-mathematical experience.
2. (Therefore) The problem originates with the student, and
3. The goal of the activity is understanding, not a solution.

5 Scaling up to K-12

It should seem obvious that implementing research modeling experiences in K-12 education is desirable, both as a tool for recruiting students to STEM fields and training tool for preparing students for work in these fields. However, with the current state of teacher training, it is not feasible beyond a limited scale. There are small-scale programs that work on exposing K-12 students to modeling experiences, including SUMS, the St. Laurence County Mathematics Partnership at Clarkson, the Center for Connected Learning at Northwestern, or StarLogo at MIT. However these programs revolve around the expertise of a Ph.D. applied mathematician or graduate student doing the leading and the mentoring. There seem to be three barriers to implementation modeling research activities on a broad scale in K-12: First, students are not mathematically prepared to engage in these types of activities; second, teachers are not mathematically agile enough to mentor them; and third, teachers do not have the experience to even imagine these sorts of activities in the first place.

5.1 Preparing MTBI/SUMS students for research

All REUs prepare students to do research, and an examination of these programs can reveal ways to prepare students to do research that can be scaled up to a larger population. Again, we use MTBI/SUMS as an example.

In order for a reversal of hierarchy to be successful, students must first be in a position where satisfying a curiosity mathematically seems natural. Students must be mathematically prepared so that they can be successfully inspired to think of the types of curiosities that are well satisfied by a mathematical approach, and so that they can pursue those curiosities with enough mathematical competence that mentoring is feasible.

The lecture portion of MTBI/SUMS makes use of a third type of modeling activity. In the middle of the spectrum between pesudocontext and research modeling lies the modeling exercise, exemplified by these problems from the MTBI and SUMS homework assignments:

1. (SUMS 2014) Using your favorite method find all equilibria of the model (you can assume λ is a positive constant) :

$$P_{t+1} = \frac{\lambda P_t}{(1+P_t)^2}$$

2. (SUMS 2014) Say we are studying a nonlinear model made up of predators (P) and prey (Q), where $-sPQ$ denotes the deleterious effect P has on Q, and kPQ represents the positive effect of Q on P. What is the biological interpretation of the assumption that $s > k$?

3. (MTBI 2014) Limpets and seaweed live in a tide pool. The dynamics of this system are given by the differential equations

$$\frac{ds}{dt} = s - s^2 - sl$$

$$\frac{dl}{dt} = sl - \frac{l}{2} - l^2$$

$$l \geq 0, s \geq 0$$

(a) Determine all equilibria of this system.

(b) For each nonzero equilibrium determined in part (a) evaluate the stability and classify it as a node, focus, or saddle point.

(c) Sketch the flows in the phase plane.

These modeling exercises have some similarity to pseudocontext tasks. Similar problems — in which a context is provided along with an equation and then the student is asked to work only with the equation — are a staple of "application" problems in nearly every textbook. However, MTBI/SUMS modeling exercises are demonstrably successful in preparing students for research modeling while other superficially similar exercises are not.

Unlike the research modeling activities described above, these are much more structured exercises that target the development of specific mathematical tools used in studying dynamical systems. Although the models provided in these examples have biological meaning, the student is not always asked to interact with the problem in context. In the above examples, the student is only required to use the context in Example 2. Examples 1 and 3 only require that the student interact with the task mathematically. The context could be completely excluded and the task could still be solved. We suspect that the differences between these exercises and pseudocontect are not so much in the individual exercises, but rather in their use: their place in the larger MTBI/SUMS program.

Every student in MTBI/SUMS knows that the session will end with an assignment to do their own project. In this scenario,these modeling exercises are embedded in take on a different significance. While the context may not be mathematically necessary to solve the exercise in front of them, it is necessary to prepare them for their projects. The context becomes an example of the types of modeling situations that can be addressed with this particular mathematical approach. Students know that they will need these examples to pursue their own projects. So it is not the individual exercise that is important for the goal of teaching a modeling persepctive, but rather the accumulation of exercises that show what types of problems can be addressed mathematically, and what different types of scenarios lend themselves to different approaches.

In addition to situational context, MTBI/SUMS modeling exercises must also be examined in their mathematical context. An isolated exercise may just be about applying a particular mathematical technique, such as finding a Jacobian. In the larger context, however, exercises are selected so that students will experience critical distinctions in dynamical systems: between discrete and continuous, linear and nonlinear, deterministic and stochastic. Each distinction affects both the behavior of the system, the degree to which the system can be explored, and the tools used to explore it. It is in this area that teacher training is lacking. They rarely have the experiences necessary to emphasize these distinctions in their own mind.

5.2 Attending to teacher training

Teachers tend to teach mathematics in the way that they were taught, and the mathematics that teachers learn in school and teach in school is deficient for preparing students for mathematical research. Exploring the ways in which REUs prepare students for mathematical research can highlight areas of school mathematics that need changing, and areas of teacher training that can be improved. Consider the following examples based on MTBI's highlighted distinctions: between discrete and continuous, linear and nonlinear, deterministic and stochastic

5.2.1 Discrete and continuous

The distinction between discrete and continuous has a huge impact on the behavior of dynamical systems. While a one dimensional discrete system such as $P_{n+1} = rP_n(1 - P_n)$ can exhibit chaotic behavior, chaos is impossible in continuous systems of fewer than three dimensions, no matter how complex. However, work with students has shown that the distinction between discrete and continuous can be very difficult for students who are used to exercises in plotting points and connecting the dots [29]. Even highly successful students will show a preference for whole number counting and regularly spaced rational numbers that interfere with their ability to draw conclusions about continuous systems [9, 10, 12], and secondary teachers suffer from similar interference from discrete thinking [45].

5.2.2 Linear and nonlinear

In traditional K-12 schooling "linear" is a modifier that describes primarily functions or graphs of functions, so linear means "straight" and nonlinear means "curved." The meaning of linear becomes slightly extended when classes begin to discuss "systems of linear equations," but a "linear equation" really only means that each variable can be expressed as a linear function of the others, that is, "linear," in this sense, is still modifying function. In general, K-12 students and teachers deal with models that depend on linear concepts and so learning the difference between linear and nonlinear problems is challenging particularly at the K-12 level.

Robert May [30], uses "linear" as a modifier not only for "linear functions" or "linear equations," but also for "linear systems" and "linear problems," a critical distinction when the goal is to study dynamics. May distinguishes between linear or nonlinear systems of differential equations and linear or nonlinear difference equations — the principle of superposition being in general lost in the nonlinear world. For example, in the equation $X_{t+1} = aX_t$, X is a nonlinear (more precisely geometric) function of t, but the equation is a "linear equation" because X_{t+1} is a linear function of X_t; the model is based on linear concepts. In contrast $X_{t+1} = aX_t(1 - X_t)$ (*), "arguably the simplest interesting nonlinear difference equation," with X_{t+1} a nonlinear function of X_t, is not based on linear concepts and as a consequence, here we lose superposition. May sees this distinction as critical saying:

> The elegant body of mathematical theory pertaining to linear systems (Fourier analysis, orthogonal functions, and so on), and its successful application to many fundamentally linear problems in the physical sciences, tends to dominate even moderately advanced University courses in mathematics and theoretical physics. The mathematical intuition so developed ill equips the student to confront the bizarre behaviour exhibited by the simplest of discrete nonlinear systems, such as equation [(*)]. Yet such nonlinear systems are surely the rule, not the exception, outside the physical sciences.[30]

In his classic *Real and Complex Analysis*, Walter Rudin introduces the exponential function as the most important function in mathematics [39, p1]. The exponential family is typically defined as the unique solution of the linear problem $\frac{dx}{dt} = ax$, $x(0) = x_0$, that is, when the rate of change of a function is proportional to the value of the function itself [4]. The solution of this linear differential equation is by definition the exponential function. Part

[4]Generalizations include, for example, $\frac{dX}{dt} = AX$ where A is an nxn matrix and X is an nx1 vector

of the importance of the exponential function lies on its ability "locally" approximate nonlinear systems; no different than using tangent planes to approximate surfaces locally, a process referred to as "lineararization" or "linear analysis" [30], intimately connected to the principle of superposition.

Robert May [30] has suggested that this distinction between linear or nonlinear systems or problems (concepts based on the study of dynamics) be introduced to students as early as possible in their education, before calculus. Examples such as SUMS show that this suggestion can be implemented realistically, and further experiments show that the linear property (rate proportional to amount) used to define an exponential function can be realistically introduced before calculus as well [10, 44].

5.2.3 Deterministic and stochastic

Lastly, stochastic processes (such as agent based models) have been shown to have tremendous applications in helping students develop a research modeling mindset [34]. Because the high degree of complexity can be managed by computer experiment, a little programing training allows students to explore complex systems they would not be mathematically quipped to handle with algebraic tools. They can be thought of as a "gateway drug" into mathematical modeling: a gentle beginning that encourages students to try harder math later on. Agent-based models and Markov chains (both discrete and continuous) are popular techniques among the students of both MTBI and SUMS.

5.2.4 Suggestions

Here, REUs can have a social impact by inviting mathematics education researchers to observe and study. Mathematics education researchers can take what they learn about effective preparation for research from REUs and use these results to improve teacher-training programs, or to inspire further research into high school student learning. In fact, much of the research cited above [8, 9, 10, 12] was born from collaborations between a mathematics education researcher and MTBI. This example shows the possibility of such fruitful collaboration [5].

[5] For more on the role of example cases in science, see [23]

5.3 Attending to teacher's experiences

However, the greatest barrier to the implementation of mathematical modeling research activities in schools is simply that very few teachers can imagine them. Teachers rarely attend REUs and rarely attend graduate school in applied mathematics. The simple fact is, the vast majority of people charged with teaching mathematical modeling have never done it.

Without some sort of experience in mathematical modeling, teachers cannot imagine what it is, let alone how to implement it in their own classrooms. Instead, they are left to rely on the experiences that they do have: textbook exercises and pseudocontext examples in which the applications are informed, but not genuine or meaningful. This cycle then perpetuates itself as future teachers grow up and learn in exactly the same limited environment.

Here again, REUs can have an impact. Teachers are trained as undergraduates. Secondary teachers in particular receive extensive mathematical training that qualifies them for REUs, and REU programs can provide exactly the mathematical research experience that teachers need in order to begin imagining a different sort of task for their students. REUs do not generally recruit from this pool because teachers are rarely interested in graduate study, and not all REUs will be interested or even appropriate for this sort of work; but for REUs that are interested in effecting social change, recruiting qualified undergraduate mathematics education majors is one possible way to having a large impact.

6 Recommendations and Future Work

REUs have tremendous potential for social impact. MTBI is one example of such a program, one which has been tremendously successful because of its social agenda. Not all REU directors will be interested in taking this route, and that is understandable. But for REU directors who wish to use their program effect social change, we have some recommendations. Improving K-12 education is an area of high need, and this is an area where social impacts can be made with only small changes to an REU program, or more sweeping changes if desired.

First: build collaborations with mathematics education faculty. Collaborations between MTBI/SUMS and mathematics education faculty have been fruitful, resulting in improved understanding of how the REU operates, well received mentorship workshops for REU faculty and graduate students, and improvements to teacher training programs run by the mathematics education faculty. Building a collaboration with a mathematics education researcher need not be elaborate or time consuming. It is simply a matter of issuing an invitation to the right person.

Second: recruit secondary mathematics education majors. Future teachers can have a huge impact. A successful teacher will teach upwards for 5000 students in their lifetime. Creating good mathematical modeling experiences for 5000 students improves mathematical citizenship for everyone, and increases interest in STEM major programs. Recruiting and contributing to the training of several future teachers has the potential to dramatically change the landscape that undergraduate programs recruit from. Here again, we do not suggest that REUs make dramatic changes to their program. Secondary mathematics education majors are qualified for these programs by their coursework, and adjusting the curriculum would eliminate the authenticity of the research experience. Insteady we suggest that REU faculty simply make an effort to invite these students (particularly strong students that the faculty personally encounter in their teaching), and market to the demographic. REU development does not happen all at once. Similar to the way MTBI has changed over decades, programs can scale up as the needs of these students are better understood through experience.

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